# EXTREMALLY DISCONNECTED
# $S$-SPACES

by

Michael L. Wage

# EXTREMALLY DISCONNECTED S-SPACES

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Recently there has been much work done on S-spaces and L-spaces (see [K], [JKR], and [R] for example).  Examples of S-spaces have previously been constructed under diamond by Ostaszewski and under CH by Hajnal, Juhász, Kunen, and Rudin, but no absolute example has been constructed.  In this note we give, under club, an example of an extremally disconnected S-space and prove that such spaces are normal and, in many cases, perfect.  The addition of extremal disconnectivity not only forces S-spaces to be nice, but also seems to simplify their construction.

*Definitions and notation.*  A space is called *extremally disconnected* if the closure of every open set is open.  This is equivalent to saying that disjoint open sets have disjoint closures X is called an *S-space* if X is regular, hereditarily separable, but not Lindelöf.  X is a *strong S-space* if each finite power of X is an S-space.  We use the term *perfect* to mean that each closed subset is a $G_\delta$.  $A^-$ and $A^c$ denote, respectively, the closure and complement of A.  Two subsets of a space, A and B, are called *separated* if $A^- \cap B = \phi = A \cap B^-$.  The only set theoretic assumption used in this note is club, an axiom that is independent of and consistent with the usual axioms for set theory.  Let $\Lambda$ be the set of all limit points in $\omega_1$.  Club states that there exists a collection $\{S_\lambda \mid \lambda \in \Lambda\}$ such that

a) for all $\lambda \in \Lambda$, $S_\lambda$ is a cofinal subset of $\lambda$ that has finite intersection with each ordinal less than $\lambda$, and

b) if C is an uncountable subset of $\omega_1$, then there exists a $\lambda$ such that $S_\lambda \subset C$.

   *Theorem 1.  Each extremally disconnected S-space is
hereditarily normal.*

   *Proof.*  Let H and K be separated (see the definition above)
subsets of an extremally disconnected S-space X.  Since X is
hereditarily separable, there exist countable sets $D \subset H$ and
$E \subset K$ such that D is dense in H and E is dense in K.  D and E
are countable separated subsets of a regular space and hence can
be separated by disjoint open sets, $U \supset D$ and $V \supset E$.  The
extremal disconnectivity of X implies that $U^-$ and $V^-$ are two
disjoint open sets that separate H from K.  It follows that
each pair of separated sets can be separated by disjoint open
sets, and thus X is hereditarily normal.

   *Example.*  Although each regular hereditarily separable
extremally disconnected space is normal, it is not true that
each Hausdorff hereditarily separable extremally disconnected
space is regular.  An easy example is the following:  Let
$N = \{(n,m) \mid n,m \leq \omega\} - \{(n,\omega) \mid n < \omega\}$ and fix a free ultrafilter, F,
on $\omega$.  Declare each point of $\{(n,m) \mid n,m < \omega\}$ to be open.  For each
$f \in F$ and $m < \omega$, a basic open set around the point $(\omega,m)$ is de-
fined to be $\{(\omega,m)\} \cup \{(n,m) \mid n \in f\}$.  For each $g \in F$ and each
sequence $\{f_m \mid m < \omega\} \subset F$, we define $\{(\omega,\omega)\} \cup \{(n,m) \mid n \in f_m, m \in g\}$ to
be a basic open neighborhood of the point $(\omega,\omega)$.

   *Lemma.*  *Let X be an extremally disconnected S-space.  If H
is closed in X, U is open in X, and $H \cap U$ is dense in H, then
$H \cup U$ is an open set.*

   *Proof.*  Suppose to the contrary that X, H, and U are as in
the hypothesis of the lemma but $H \cup U$ is not open.  Then there
is a point $h \in H$ such that h is in the closure of $X - (H \cup U)$.
Let D be a countable dense subset of $H \cap U$ and E be a countable
dense subset of $X - (H \cup U)$.  Notice that D and E are countable
separated subsets of X and thus can be separated by disjoint

open sets.  But h is in the closure of both D and E, and

hence is in the closure of the disjoint open sets that separate

them.  This contradicts the extremal disconnectivity of X and

finishes the proof of the lemma.

   *Theorem 2.   If X is an extremally disconnected S-space in*

*which each countable set is a* $G_\delta$, *then X is perfect.*

   *Proof.*   Let H be a closed subset of X and let D be a

countable dense subset of H.  By assumption, there exist open

sets, $U_n$, $n < \omega$, such that $D = \bigcap \{U_n | n < \omega\}$.  It follows from the

lemma that for each n, $U_n \cup H$ is an open set.  Then

$H = \bigcap \{U_n \cup H | n < \omega\}$ and hence is a $G_\delta$.

   This theorem is especially interesting since most of the

spaces constructed using Ostaszewski's technique of building

spaces by induction are locally countable, and hence have counta-

ble subsets being $G_\delta$.

   The following related theorem has been proved by A. Berner

and the author in [W].

   *Theorem 3.   There is no extremally disconnected hereditarily*

*separable Dowker space.*

   *Example.*   Club implies there exists an extremally dis-

connected S-space.  Let $M = \omega_1$.  We will inductively define

topologies, $\tau_\lambda$, on initial segments, $\lambda$, of M.  Recall that

$\{S_\lambda | \lambda \in \Lambda\}$ is the sequence associated with club.  Let $\tau_\omega$ be the

discrete topology on $\omega$.  Suppose $\lambda \in \Lambda$ and we have defined $\tau_\xi$

on $\xi$ for each limit ordinal $\xi < \lambda$ such that

   1) $\tau_\xi$ is $T_3$ and extremally disconnected,

   2) $\tau_\gamma = \tau_\xi \cap P(\gamma)$ and $\eta \in \tau_\xi$ for all $\eta, \gamma < \xi$ such that $\gamma$
      is a limit, and

   3) $\eta \in cl_{\tau_\xi}(S_\gamma)$ for all $\gamma \leq \eta < \xi$ such that $\gamma$ is a limit.

We must show how to define $\tau_\lambda$ on $\lambda$ so that (1)-(3) hold for

all limits $\xi \leq \lambda$.

If $\lambda$ is a limit of limits, let $\tau_\lambda$ be the topology generated by the base $\bigcup_{\xi < \lambda} \tau_\xi$. Conditions (2) and (3) above are then easily verified for each limit $\xi \leq \lambda$ as is extremal disconnectivity. Regularity is a little trickier. Suppose $\alpha \in U$, an open set in $(\lambda, \tau_\lambda)$. To show that $(\lambda, \tau_\lambda)$ is regular, we will find a $V \subset U$ that is clopen in $(\lambda, \tau_\lambda)$ and contains $\alpha$. Without loss of generality, $U \subset \alpha+1$ by condition (2). Let $\xi = \alpha+\omega$. Then since $\tau_\xi$ is regular and extremally disconnected, there exists a $V$, clopen in $\tau_\xi$, with $\alpha \in V \subset U$. $V$ is open in $\tau_\lambda$, but we must show it is clopen in $\tau_\lambda$ to complete the proof of regularity. Consider $\xi-V$. Since $V$ is closed in $\tau_\xi$, $\xi-V$ is open in $\tau_\xi$, and hence, by condition (2), open in $\tau_\lambda$. But $\xi-V$ contains all but finitely many points of $S_\xi$ (since $\xi = \alpha+\omega$ so that $S_\xi$, by definition, has finite intersection with $\alpha+1$, and hence with $V$) and by condition (3), $\lambda-\xi \subset \text{cl}_{\tau_\lambda}(S_\xi)$, so $\text{cl}_{\tau_\lambda}(\xi-V) = \lambda-V$. Since in an extremally disconnected space, disjoint open sets have disjoint closures, $V$ must be closed in $\tau_\lambda$, and the limit of limits case is completed.

Now suppose $\lambda = \gamma+\omega$ for some limit $\gamma$. We must define neighborhoods of the points $\gamma+n$ for each $n < \omega$. Partition $S_\gamma$ into $\omega$ infinite pieces, $W_n$, and fix a free ultrafilter $F_n$ on $W_n$ for each $n < \omega$. A basic neighborhood of $\gamma+n$ is defined to be any set of the form $\{\gamma+n\} \cup U$ where $U \in \tau_\gamma$ and $U$ contains some member of $F_n$. Let these basic neighborhoods, together with $\tau_\gamma$, be a base for $\tau_\lambda$. With this definition it is easy to check (1)-(3) for each limit $\xi \leq \lambda$. Regularity follows from the fact that $\tau_\gamma$ is normal (it is Lindelöf and regular) and $S_\gamma$ is closed discrete in $\tau_\gamma$ (by condition (2)).

Finally, we let $\tau = \tau_{\omega_1}$ be the topology on M. We have already shown that M is extremally disconnected and regular and condition (3) guarantees that M is an S-space. Together,

Theorems 1 and 2 show that M is perfect and hereditarily normal.

*Remarks.* A slight modification of M shows that not every extremally disconnected S-space is perfect: Let $M^* = M \cup \{\omega_1\}$. Topologize $M^*$ by giving M the usual topology, $\tau$, and letting the basic neighborhoods of the point $\omega_1$ be $\{\omega_1\} \cup U$ where U is any co-countable member of $\tau$. Then $M^*$ is an extremally disconnected S-space but is not perfect since the point $\omega_1$ is not a $G_\delta$.

The space M is not realcompact. Ken Kunen noticed that a realcompact extremally disconnected S-space can be constructed by using the above technique if the construction is done on a Souslin tree. Kunen also noticed that $M^2$ is not hereditarily separable and hence M is not a strong S-space. The existence of an extremally disconnected strong S-space is still an open question.

### References

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