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Our study consists of two parts. In the first part we determine very general conditions under which the following concepts are equivalent: (i) f is a closed map, (ii) f is a compact map, (iii) f is a quotient map and (iv) f is a compact-covering map. These results not only improve known results but also settle a variety of appealing conjectures.

In the second part we study the preservation of topological properties by monotone maps. These results improve some results of Hanai.

1. Behavior of Monotone Functions

Many a beautiful and important theorem concerning the preservation of topological properties by continuous functions has been proved. Generally, one must consider continuous functions f which satisfy additional properties. The following properties are very frequently used:

(a) f is a *closed* map,

(b) f is a *quotient* map,

(c) f is *compact* (i.e. $f^{-1}(C)$ is compact whenever C is compact),

(d) f is *compact-covering* (i.e. for each compact set B there exists a compact set A such that $f(A) = B$).

These four concepts are surprisingly different. However, under surprisingly mild restrictions, they are also equivalent. Indeed we can easily prove the following results:

Theorem 1.1. Let $f:X \to Y$ *be a continuous monotone (i.e.* $f^{-1}(p)$ *is connected for each* $p \in Y$*) map from the locally peripherally compact Hausdorff space* X *onto the Hausdorff k-space* Y.

If $f^{-1}(p)$ *is compact for each* $p \in Y$, *then the following are equivalent:*

(a) f *is compact,*

(b) f *is a closed map,*

(c) f *is a quotient map,*

(d) f *is compact-covering.*

Proof. It is well-known that (b) implies (a) (See, for example, the introduction of Whyburn [9]). Also, by Lemma 3.4 of [1], (c) implies (b). Lemma 11.2 of [6] proves that (d) implies (c). Clearly (a) implies (d), which completes the proof.

Theorem 1.2. Let $f:X \to Y$ *be a continuous monotone map from the locally peripherally compact Hausdorff space* X *onto the Hausdorff k-space* Y. *If bdry* $f^{-1}(p)$ *is compact for each* $p \in Y$, *then the following are equivalent:*

(a) f *is a quotient map,*

(b) f *is a closed map,*

(c) f *is a compact-covering map.*

Proof. By Lemma 3.4 Of [1], (a) implies (b). To see that (b) implies (c), let $X' = X - \bigcup\{[f^{-1}(p)]^{\circ} | p \in Y\}$ and note that $f|X'$ is closed and has compact point inverses. Therefore $f|X'$ is compact (see introduction of [9]) which implies that f is compact-covering, as required. Lemma 11.2 of [6] proves that (c) implies (a). (Indeed this last implication is the only one which requires that Y be a k-space!)

Note that Hausdorff k-spaces are exactly the quotient spaces of locally compact Hausdorff spaces (see Theorem 9.4 on p. 248 of [4]). Therefore, Theorem 1.2 automatically yields substantial improvements of Corollaries 2.61 and 2.62 of [9] and of Theorem 9 of [5].

The following two simple examples clearly show that none of the hypotheses of Theorems 1.1 and 1.2 are superfluous.

Example 1.3. There exists a σ-compact metric space X, *a compact metric space* Y *and a compact-covering quotient map* f *from* X *onto* Y *such that*

(a) $f^{-1}(p)$ *is compact and connected for each* $p \in Y$.

(b) f *is not closed (and hence not compact, by Theorem 1.2).*

Proof. Let E be the euclidean plane with the usual topology and let $A_n = \{(np,p) \in E | n^{-2} \leq p \leq n\}$, for each positive integer n (i.e. A_n is a certain line segment of the line $x = ny$). We now let X and Y be the following subspaces of E:

$X = \{(0,0)\} \cup \bigcup_{n=1}^{\infty} A_n$, $\qquad Y = \{(0,0)\} \cup \{(n^{-2},n^{-1})\}_{n=1}^{\infty}$.

Finally we define $f:X \to Y$ by $f(0,0) = (0,0)$, $f(A_n) = (n^{-2},n^{-1})$, for each n. All our requirements are clearly satisfied.

Example 1.4. There exists a locally compact metric space X, *a compact metric space* Y *and a quotient map* f *from* X *onto* Y *such that*

(a) $f^{-1}(p)$ *is compact for each* $p \in Y$,

(b) f *is not closed (and hence not compact, by Theorem 1.2).*

Proof. Exactly the same as the proof of Example 1.3, except that $A_n = \{(np,p) \in E | p = n^{-2} \text{ or } n^{-1} \leq p \leq n\}$, for each positive integer n.

2. Monotone Quotient Images

We will prove the following result, which is an improvement of Theorem 9 in Hanai [6]:

Theorem 2.1. If $f:X \to Y$ *is a monotone quotient map from a metrizable locally separable (locally compact) space* X *onto a regular first countable space* Y, *then* Y *is a metrizable locally separable (locally compact) space.*

Surprisingly, this result becomes false if "metrizable" is everywhere replaced by "stratifiable" (see Example 2.4).

Indeed we will first prove the stronger result.

Theorem 2.2. If $f: X \to Y$ *is a monotone quotient map from the paracompact locally Lindelöf (locally separable) space* X *onto the regular space* Y, *then* Y *is paracompact and locally Lindelöf (locally separable).*

Proof. We will prove this result for a paracompact locally Lindelöf space X (for paracompact locally separable spaces the proof is similar and simpler): Let $\mathcal{U}$ be a locally finite open cover of X such that the closure of each element of $\mathcal{U}$ is Lindelöf and let $\mathcal{V}$ be an open cover of X such that $\{st(st(x,\mathcal{V}),\mathcal{V}) \mid x \in X\}$ is a refinement of $\mathcal{U}$.

For $V \in \mathcal{V}$, let $V_0 = st(V,\mathcal{V})$ and, for each integer $n \geq 1$, let $V_n = st(V_{n-1},\mathcal{V})$. Then the set $V_* = \bigcup_{n=1}^{\infty} V_n$ is clopen (i.e. open and closed). (Clearly V_* is open. Let $x \in V_*^-$; then $st(x,\mathcal{V}) \cap V_* \neq \emptyset$ and thus $st(x,\mathcal{V}) \cap V_n \neq \emptyset$ for some n; hence $x \in V_{n+1} \subset V_*$. Thus V_* is closed.) Furthermore V_* is Lindelöf: Since V^- is Lindelöf and $\{st(x,\mathcal{V}) \cap V^- \mid x \in V^-\}$ is an open cover of V^- there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points of V^- such that $V^- \subset \bigcup_{n=1}^{\infty} st(x_n,\mathcal{V})$. Thus $V_o^- = st(V,\mathcal{V})^-$ $\subset st(\bigcup_{n=1}^{\infty} st(x_n,\mathcal{V}),\mathcal{V})^- = (\bigcup_{n=1}^{\infty} st(st(x_n,\mathcal{V}),\mathcal{V}))^-$. On the other hand, there exists a $U_n \in \mathcal{U}$ such that $U_n^- \supset st(st(x_n,\mathcal{V}),\mathcal{V})^-$ for each n. Hence $V_o^- \subset \bigcup_{n=1}^{\infty} U_n^-$ is Lindelöf. By induction, V_n^-(n=0, 1, 2, ...) is Lindelöf. Since $V_* = \bigcup_{n=o}^{\infty} V_n^-$, we get that V_* is Lindelöf.

Thus, by the construction of V_*, we easily see that X is covered by a pairwise disjoint family $\{X_\alpha\}_{\alpha \in A}$ of clopen Lindelöf subsets.

We show $\{f(X_\alpha)\}_{\alpha\in A}$ is a disjoint family of clopen Lindelöf subsets of Y which covers Y: Clearly each $f(X_\alpha)$ is Lindelöf and $\{f(X_\alpha)\}_{\alpha\in A}$ covers Y. Since f is monotone, $X_\alpha = f^{-1}(f(X_\alpha))$ for each $\alpha \in A$ and thus $\{f(X_\alpha)\}_{\alpha\in A}$ is a disjoint family of subsets of Y. Since f is a quotient map and $X_\alpha = f^{-1}(f(X_\alpha))$, each $f(X_\alpha)$ is clopen.

Therefore Y is the union of a pairwise disjoint family of open regular Lindelöf (hence, paracompact) subsets. This completes the proof.

Proof of Theorem 2.1. By the method of proof of Theorem 2.2 we get that X is covered by a pairwise disjoint family $\{X_\alpha\}_{\alpha\in A}$ of clopen separable subsets. Thus, by the Corollary on page 695 of [8], Y is covered by the family $\{f(X_\alpha)\}_{\alpha\in A}$ of open separable metrizable subsets. Hence Y is clearly metrizable and locally separable. The "local compactness" follows from Lemma 1 in [8].

Example 6.1 in Stone [8] shows that Theorem 3.1 is false if f is not monotone, and the following example shows that Theorem 2.1 is false if X is not locally separable. Thus none of the hypothesis in Theorem 2.1 is superfluous.

Example 2.3. There exist topological spaces X *and* Y *such that* X *is metrizable and not locally separable,* Y *is hereditarily paracompact, first countable and not metrizable, and an onto monotone quotient map* $f:X \to Y$ *with* $f^{-1}(y)$ *compact for each* $y \in Y$ *(furthermore,* f *is pseudo-open).*

Proof. Let the set of irrational numbers $A = \bigcup_{n=1}^{\infty} A_n$ such that the A_n are uncountable, disjoint and dense in the euclidean line. (This can easily be done!) Let

$X = \{(x,y) | 0 \leq y \leq 1,\ y \geq 1/n$ for $x \in A_n,\ y = 0$ for x rational$\}$, with the following topology: A neighborhood of $(x,0) \in X$ is the intersection of X with an open disk, in the plane, centered about $(x,0)$; a neighborhood of $(x,z) \in X$, with x irrational, is an open interval (intersected with X) of the vertical line $x = z$ containing (x,z).

Let Y be the set of real numbers and define $f:X \to Y$ by $f(x,w) = X$, for each $(x,w) \in X$. Give Y the quotient topology with respect to f. It is easily seen that all our claims are satisfied.

Example 2.4. There exist separable first countable topological spaces X *and* Y *such that* X *is an* M_1*-space,* Y *is not monotonically normal, and an onto monotone quotient map* $f:X \to Y$ *such that each* $f^{-1}(y)$ *is compact.*

Proof. (We modify an example of van Douwen [3].) Let $Z = P \cup Q$, where $P = \{(x,0) | x \text{ irrational}\}$ and $Q = \{(x,y) | x,y \text{ rational},\ y > 0\}$. Let $(x,y) = n_{xy}$ be a one-to-one correspondence between Q and $\{1 - 1/n | n \text{ is a positive integer}\}$.

Let Y be the set Z with the following topology: A neighborhood of $p = (x,0)$ is of the form

$$B(p,\varepsilon) = \{(s,t) \in Z | t \leq |x - s| < \varepsilon\}$$

for any $\varepsilon > 0$. A neighborhood of $(x,y) \in Q$ is an ordinary euclidean neighborhood. In Example 2.4 of [3], it is proved that Y is not monotonically normal.

Let $X = Z \times I$, where I is the closed unit interval. Define a topology on X, as follows: A neighborhood of $((x,0),0) = (p,0)$ is of the form $S(p,\varepsilon) = B(p,\varepsilon) \times [0, 1/n]$, for some positive integer n and $\varepsilon > 0$.

A neighborhood of $((x,y),n_{xy})$ with x,y rational, is of the form $U \times V$ such that U is an ordinary euclidean neighborhood of (x,y) and V is an open interval (intersected with I) centered at n_{xy}. A neighborhood of any other ((x,y),t) is of the form $\{(x,y)\} \times N_t$ such that N_t is an open interval centered at t (intersected with I). It is easily seen that X is an hereditarily M_1-space, because of Example 2.3 of [3]. Clearly, both X and Y are separable first countable spaces.

Let $f:X \to Y$ be defined by $f((x,w),t) = (x,w)$. It is easily seen that f is a monotone quotient map.

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