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## STRATIFIABLE SPACES ARE $M_2$

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## STRATIFIABLE SPACES ARE $M_2$

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### 1. Introduction

The  $M_i$ -spaces,  $i = 1, 2, 3$ , were defined in 1961 by J. Ceder [3] as natural generalizations of metrizable spaces. C. R. Borges [1] later showed that the class of  $M_3$ -spaces, which he renamed "stratifiable" spaces, was a particularly well-behaved class of spaces. He showed, for example, that stratifiable spaces are closed under closed maps, and that the Dugundji extension theorem for metrizable spaces could be generalized to stratifiable spaces. Since then, stratifiable spaces and some of their generalizations (such as semi-stratifiable spaces and monotonically normal spaces) have played an important role in general topology.

From the definitions of the  $M_i$ -spaces, it is clear that  $M_1 \Rightarrow M_2 \Rightarrow M_3$ . The major question for  $M_i$ -spaces has been whether any of the implications reverse. A giant step in this direction came in 1973 when Heath [4] proved that stratifiable spaces have a  $\sigma$ -discrete network. We prove in this paper, with the help of Heath's result, that  $M_3$ -spaces are the same as  $M_2$ -spaces.<sup>1</sup> We also give two specific classes of spaces for which  $M_3 \Rightarrow M_1$ .

### 2. Definitions and other preliminaries

A collection  $\mathcal{K}$  of subsets of a space  $X$  is *closure-preserving* if whenever  $\mathcal{K}' \subset \mathcal{K}$ , then  $Cl(\cup \mathcal{K}') = \cup \{\bar{H} \mid H \in \mathcal{K}'\}$ . A pair-collection  $\mathcal{Q} = \{P = (P_1, P_2) \mid P \in \mathcal{Q}\}$  is *cushioned* if whenever  $\mathcal{Q}' \subset \mathcal{Q}$ , then  $Cl(\cup \{P_1 \mid P \in \mathcal{Q}'\}) \subset \cup \{P_2 \mid P \in \mathcal{Q}'\}$ .

A collection  $\mathcal{B}$  is a *quasi-base* for  $X$  if whenever  $x \in U$  and  $U$  is open, there exists  $B \in \mathcal{B}$  such that  $x \in \text{Int}(B)$  and  $B \subset U$ . A pair-collection  $\mathcal{B} = \{B = (B_1, B_2) \mid B \in \mathcal{B}\}$  is a *pair-base* for

(1) After it was announced at the Auburn conference, the author learned that Heikki Junnila had also obtained this result.

$X$  if the  $B_1$ 's are open, and whenever  $x \in U$  and  $U$  is open, there exists  $B \in \mathcal{B}$  such that  $x \in B_1 \subset B_2 \subset U$ .

A space  $X$  is an  $M_1$ -space ( $M_2$ -space) if  $X$  has a  $\sigma$ -closure-preserving base (quasi-base).  $X$  is an  $M_3$ -space (or stratifiable space) if  $X$  has a  $\sigma$ -cushioned pair-base.

A collection  $\mathcal{F}$  is a network for a space  $X$  if whenever  $x \in U$  and  $U$  is open, there exists  $F \in \mathcal{F}$  such that  $x \in F \subset U$ .  $X$  is a  $\sigma$ -space if  $X$  has a  $\sigma$ -discrete network.

A space  $X$  is monotonically normal if for every pair of disjoint closed sets  $H$  and  $K$ , there exists an open set  $D(H,K)$  containing  $H$  such that  $\overline{D(H,K)} \cap K = \emptyset$ , and if  $H' \supset H$  and  $K' \subset K$ , then  $D(H',K') \supset D(H,K)$ .

In our proofs, we shall make use of the following known results:

(1) Stratifiable spaces are hereditarily paracompact and monotonically normal [1], [3], and also [5].

(2) Stratifiable spaces are  $\sigma$ -spaces [4].

(3) Stratifiable spaces are  $M_1(M_2)$  if and only if whenever  $X$  is stratifiable and  $p \in X$ , then  $p$  has a  $\sigma$ -closure-preserving base (quasi-base) [2].

We shall make frequent use of the following notation. If  $F \subset V \subset X$ , where  $F$  is closed and  $V$  is open in the stratifiable space  $X$ , let  $V_F = D(F, X - V)$ , where  $D$  is a monotone normality operator on  $X$ . If  $n \in \mathbb{N}$ , and  $V_F^n$  is defined, let  $V_F^{n+1} = D(F, X - V_F^n)$ . If  $p$  is a point in  $V$ , let  $V_p^n = V_{\{p\}}^n$ . Note that if  $F \subset G$  and  $V \subset W$ , and  $W_G$  is defined, then  $V_F^n \subset W_G^n$  for all  $n \in \mathbb{N}$ .

### 3. Main results

*Theorem 1.* Stratifiable spaces are  $M_2$ .

*Proof.* Let  $X$  be stratifiable and let  $p \in X$ . We need only show that  $p$  has a closure-preserving quasi-base. Let

$\mathcal{F}' = \bigcup_{n=1}^{\infty} \mathcal{F}'_n$  be a  $\sigma$ -discrete network for  $X$ ; let

$\mathcal{F}_n = \{F \in \mathcal{F}' \mid p \notin F\}$ , and let  $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ . Using the hereditary paracompactness of  $X$ , it is easy to find a sequence  $\{\mathcal{U}_n\}$  of locally finite open covers of  $M = X - \{p\}$  such that

- (i) if  $x \in F \in \mathcal{F}_n$ , then  $\text{st}(x, \mathcal{U}_n) \subset (X - \overline{(X - F)}_p)$
- (ii)  $\mathcal{U}_{n+1}$  star-refines  $\mathcal{U}_n$ , and
- (iii)  $p \notin \overline{\text{st}(x, \mathcal{U}_1)}$  for all  $x \in M$ .

The sequence  $\{\mathcal{U}_n\}_{n=1}^{\infty}$  generates a metric topology  $T$  on  $M$  such that  $T$  is coarser than the original topology, and if  $x \in F \in \mathcal{F}$ , then  $x \in \text{Int}_T(X - \overline{(X - F)}_p)$ . There exists a metric  $d$  on  $M$  which generates  $T$  such that for every  $x \in M$ , it is true that  $p \notin \overline{B(x, 1)}$ , where  $B(x, n)$  is the ball about  $x$  of radius  $1/n$ .

For  $x \in M$ , let  $W(x) = \bigcup_{n=1}^{\infty} (X - \overline{B(x, n)})_{p2}$ . Now  $W(x) \subset (X - \{x\})_{p2} \subset (X - F)_p$  for some  $F \in \mathcal{F}$  containing  $x$ . This is true because  $x \notin \overline{(X - \{x\})}_p$ , and we can choose  $F$  such that  $x \in F$  and  $F \cap (X - \{x\})_p = \emptyset$ . Since  $x \in \text{Int}_T(X - \overline{(X - F)}_p)$  there exists  $n \in \mathbb{N}$  such that  $B(x, n) \cap (X - F)_p = \emptyset$ , and hence  $B(x, n) \cap W(x) = \emptyset$  also. Let  $n(x)$  be the least positive integer such that  $B(x, n(x)) \cap W(x) = \emptyset$ . Let  $M_n = \{x \in M \mid n(x) \leq n\}$ .

We claim that for every  $n \in \mathbb{N}$ ,  $M_n$  is closed in  $M$ . To see this, fix  $n$ , and suppose  $x \in M_n$ . We need to show that  $W(x) \cap B(x, n) = \emptyset$ . To this end, let  $z \in B(x, n)$ , say  $d(x, z) = 1/n - \epsilon$ ,  $\epsilon > 0$ . Let  $k \in \mathbb{N}$ . We shall show that  $z \notin (X - \overline{B(x, k)})_{p2}$ . Pick  $m > k + n$  such that  $1/m < \epsilon$ , and pick  $y \in M_n \cap B(x, m)$ . There exists  $j \in \mathbb{N}$  such that  $B(y, j) \subset B(x, k)$ . Thus  $(X - \overline{B(y, j)})_{p2} \supset (X - \overline{B(x, k)})_{p2}$ , and  $y \in M_n$  implies  $B(y, n) \cap (X - \overline{B(y, j)})_{p2} = \emptyset$ . But  $d(y, z) \leq d(y, x) + d(x, z) < 1/m + 1/n - \epsilon < 1/n$ . Thus  $z \in B(y, n)$ , so  $z \notin (X - \overline{B(x, k)})_{p2}$ , which proves the claim. Note that we have in fact proved that  $M_n$  is closed in the metric topology  $T$ .

Let  $\mathcal{G}_1, \mathcal{G}_2, \dots$  be a sequence of locally finite covers of  $(M, T)$  such that  $G \in \mathcal{G}_i$  implies  $\text{diam}(G) < 1/i$ . For each  $x \in M$ , let  $M(x) = \bigcap \{G \in \mathcal{G}_i \mid i \leq n(x), x \in G\} - M_{n(x)} - 1$ .

Note that if  $y \in M(x)$ , then  $n(y) \geq n(x)$ , and so  $M(y) \subset M(x)$ .

For each closed set  $H \subset M$ , let  $M(H) = \bigcup_{x \in H} M(x)$ , and let  $H(p) = X - M(H)$ . We claim that  $\mathcal{K} = \{H(p) \mid H \text{ closed, } p \notin H\}$  is a closure-preserving quasi-base at  $p$ .

Clearly  $\mathcal{K}$  is a closed network at  $p$ . To see that  $\mathcal{K}$  is closure-preserving, let  $\mathcal{K}' = \{H_\alpha(p) \mid \alpha \in A\}$ , and suppose  $z \notin \bigcup \mathcal{K}'$ . Then for each  $\alpha \in A$ , there exists  $x_\alpha \in H_\alpha$  such that  $z \in M(x_\alpha)$ . Hence  $M(z) \subset M(x_\alpha) \subset M(H_\alpha)$ , and so  $M(z) \cap (\bigcup \mathcal{K}') = \emptyset$ . Thus  $\mathcal{K}$  is closure-preserving.

It remains to prove that  $p \in \text{Int}(H(p))$ . We shall show that  $(X - H)_{p4} \subset H(p)$ . Suppose not. Then there exists  $z \in (X - H)_{p4} \cap M(x)$  for some  $x \in H$ . Since  $z \in M(x) \subset B(x, n(x))$ , it must be true that  $z \notin W(x)$ . Now  $z \in (X - H)_{p4} \subset (X - \{x\})_{p4} \subset (X - F)_{p3}$  for some  $F \in \mathcal{F}$ . (We can choose  $F \subset X - (X - \{x\})_p$ .) But  $(X - F)_{p3} \subset (X - B(x, n))_{p2}$  for some  $n$ . (Choose  $n$  such that  $B(x, n) \subset X - \overline{(X - F)}_p$ .) Thus  $z \in W(x)$ , contradiction.

Our next theorem was actually proved before Theorem 1; in fact, its proof led us to the proof of Theorem 1.

*Theorem 2.* Let  $X$  be stratifiable. Suppose  $X = M \cup \{p\}$ , where  $M$  is metrizable. Then  $X$  is an  $M_1$ -space.

*Proof.* By Theorem 1, there exists a closure-preserving closed quasi-base  $\mathcal{K}$  at  $p$ . Let  $\mathcal{G}_1, \mathcal{G}_2, \dots$  be a sequence of locally finite covers of  $M$  such that if  $G \in \mathcal{G}_k$ , then  $\text{diam}(G) < 1/k$ . Let  $\mathcal{G} = \bigcup_{n=1}^\infty \mathcal{G}_n$ . If  $H \in \mathcal{K}$ , let  $\mathcal{G}(H) = \{\{p\} \cup (\bigcup \mathcal{G}') \mid \mathcal{G}' \subset \mathcal{G}, H - \{p\} \subset \bigcup \mathcal{G}'\}$ , and if  $G \in \mathcal{G}'$ , then  $G \cap H \neq \emptyset$ . Let  $\mathcal{U} = \{\mathcal{G}(H) \mid H \in \mathcal{K}\}$ .

Clearly  $\mathcal{U}$  is a base at  $p$ . We claim that  $\mathcal{U}$  is closure-preserving. Suppose  $\mathcal{U}' \subset \mathcal{U}$  and  $x \notin \bigcup \{\bar{U} \mid U \in \mathcal{U}'\}$ . Let  $\mathcal{K}' = \{H \in \mathcal{K} \mid \text{there exists } U \in \mathcal{U}' \text{ such that } U \subset \mathcal{G}(H)\}$ . Let  $K = \bigcup \mathcal{K}'$ . There exists an integer  $n \in \mathbb{N}$  such that  $d(x, K) > 1/n$ . Thus there exists an open set  $V$  containing  $x$  such

that  $V \cap (\cup \{G \in \mathcal{G}_k \mid k \geq n \text{ and } G \cap K \neq \emptyset\}) = \emptyset$ , and since the  $\mathcal{G}_i$ 's are locally finite, we can choose this  $V$  such that if  $G \in \mathcal{G}_i$ ,  $i < n$ , then  $V \cap G \neq \emptyset$  if and only if  $x \in \bar{G}$ . But now it is easy to see that  $V \cap (\cup \mathcal{U}') = \emptyset$ . Thus  $x \notin \overline{\cup \mathcal{U}'}$ , and the proof is complete.

*Theorem 3. Countable stratifiable spaces are  $M_1$ .*

*Proof.* Let  $X$  be a countable stratifiable space. It is enough to show that every point of  $X$  has a closure-preserving base.

Let  $p \in X$ ; say  $X = \{p\} \cup M$ , where  $M = \{x_1, x_2, \dots\}$ . We shall show that  $p$  has a closure-preserving base. The proof is similar to that of Theorem 1, but with certain refinements we can construct the  $M(H)$ 's, and thus the  $H(p)$ 's, in such a way that they are clopen (open and closed) sets.

Using the paracompactness of  $M$ , construct a sequence  $\{\mathcal{U}_n\}_{n=1}^\infty$  of open covers of  $M$  such that

(i)  $st(x_k, \mathcal{U}_k) \subset (X - \overline{(X - \{x_k\})_{\{p, x_1, \dots, x_{k-1}\}}})$  for  $k = 1, 2, \dots$ ;

(ii)  $\mathcal{U}_{n+1}$  star-refines  $\mathcal{U}_n$  for  $n = 1, 2, \dots$ ;

(iii) for every  $x_k \in M$ ,  $st(x_k, \mathcal{U}_k) \cap \{p, x_1, \dots, x_{k-1}\} = \emptyset$ .

This sequence generates a metric topology  $T$  on  $M$  such that  $T$  is coarser than the original topology, and

$x_k \in \text{Int}_T(X - \overline{(X - \{x_k\})_{\{p, x_1, \dots, x_{k-1}\}}})$ . There is a metric  $d$  on  $M$  which generates  $T$  such that for all  $k \in \mathbb{N}$ ,

$\overline{B(x_k, k)} \cap \{p, x_1, \dots, x_{k-1}\} = \emptyset$ . Let

$W(x_k) = \cup_{n=k}^\infty (X - \overline{B(x_k, n)})_{\{p, x_1, \dots, x_{k-1}\}}$ . As in Theorem 1, let  $n(x_k)$  be the least positive integer such that

$B(x_k, n(x_k)) \cap W(x_k) = \emptyset$ , and let  $M_n = \{x \in M \mid n(x) \leq n\}$ . The proof that  $M_n$  is closed in  $T$  is similar to the proof of this fact in Theorem 1.

Since  $M$  is countable,  $(M, T)$  is strongly 0-dimensional. Let

$\mathcal{G}_1, \mathcal{G}_2, \dots$  be a sequence of locally finite covers of  $M$  by clopen sets such that  $\text{diam}(G) < 1/n$  whenever  $G \in \mathcal{G}_n$ .

Let  $n'(x_k)$  be the least positive integer such that  $\text{st}(x_k, \mathcal{G}_{n'(x_k)}) \cap M_{n(x_k)-1} = \emptyset$ . Let  $m(x_k) = \max\{n(x_k), n'(x_k)\}$ , and let  $M(x_k) = \bigcap \{G \in \mathcal{G}_i \mid i \leq m(x_k), x_k \in G\}$ . If  $y \in M(x_k)$ , then  $n(y) \geq n(x_k)$ , and so also  $n'(y) \geq n'(x_k)$ , since  $\text{st}(y, \mathcal{G}_i) \supset \text{st}(x_k, \mathcal{G}_i)$  for  $i = 1, 2, \dots, m(x_k)$ . Thus  $m(y) \geq m(x_k)$  and  $M(y) \subset M(x_k)$ .

For  $H$  closed,  $p \notin H$ , define  $M(H)$  and  $H(p)$  as in Theorem 1. To complete the proof, we shall show that  $M(H)$  is clopen. Suppose  $x_k \notin M(H)$ . Let  $x_j \in H$ ,  $j > k$ . Then  $(X - H)_{x_k^2} \subset (X - \{x_j\})_{\{p, x_1, \dots, x_{j-1}\}^2} \subset (X - \overline{B(x_j, n)})_{\{p, x_1, \dots, x_{j-1}\}^2}$  for some  $n \in \mathbb{N}$ . (Simply choose  $n$  such that  $B(x_j, n) \cap (X - \{x_j\})_{\{p, x_1, \dots, x_{j-1}\}^2} = \emptyset$ ). Thus  $(X - H)_{x_k^2} \subset W(x_j)$ , and so  $(X - H)_{x_k^2} \cap M(x_j) = \emptyset$ . Hence  $x_k \notin \text{Cl}(\bigcup \{M(x_j) \mid j > k, x_j \in H\})$ . But  $M(x_1)$  is clopen in  $(M, \mathcal{T})$ , and hence in  $X$ , for all  $i \in \mathbb{N}$ . Thus  $x_k \notin \overline{M(H)}$ , and it follows that  $M(H)$  is a clopen set.

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