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1. Introduction

Many concepts in metrization theory have either been defined or can be characterized by means of a sequence of open covers which guarantee that certain sequences have cluster points. These concepts actually occur in "pairs" with the stronger concept requiring a certain type of sequence to cluster at a particular point, whereas the weaker concept merely requires a sequence of the same sort to cluster. The purpose of this paper will be to introduce a class of spaces motivated by this "pair" occurrence, and to investigate the relationship to various other important classes of spaces.

Unless otherwise stated, no separation axioms are assumed; however, all regular spaces are assumed to be T_1. The positive integers are denoted by N. If $\mathfrak{U}$ is a cover of X then $\mathfrak{U}^* = \{St(U,\mathfrak{U}): U \in \mathfrak{U}\}$.

Let $\langle\mathfrak{U}_n\rangle$ be a sequence of open covers of a space X. Consider the following conditions on the sequence $\langle\mathfrak{U}_n\rangle$.

(A) (i) $\mathfrak{U}_1 > \mathfrak{U}_2^* > \mathfrak{U}_2 > \mathfrak{U}_3^* > \ . \ . \ . \ .$

(ii) If $x_n \in St(x,\mathfrak{U}_n)$, then the sequence $\langle x_n\rangle$ has a cluster point.

(B) If $x_n \in St^2(x,\mathfrak{U}_n)$, then the sequence $\langle x_n\rangle$ has a cluster point.

(C) If $x_n \in St(x,\mathfrak{U}_n)$, then the sequence $\langle x_n\rangle$ has a cluster point.

(D) If $\{x_i: i \geq n\} \cup \{x\} \subset U_n \in \mathfrak{U}_n$, then the sequence $\langle x_n\rangle$ has a cluster point.

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A space X with a sequence $\langle\mathcal{U}_n\rangle$ satisfying (A), (B), (C) or (D) is called an *M-space* [20], *wM-space* [18], *wΔ-space* [4] or a *quasi-complete space* [10], respectively. If, in (A), (B) and (C), we require that the sequence $\langle x_n\rangle$ clusters to x, then (A) and (B) are well-known characterizations of metrizability (at least for T_0-spaces) and (C) is clearly equivalent to the definition of a developable space. In fact, (A) is the condition of the Alexandroff-Urysohn Metrization Theorem [1], and (B) is that of the Moore Metrization Theorem [19] (see also [18, Theorem 2.3 in II]).

The conditions given in (A), (B) and (C) illustrate our previous discussion concerning certain concepts occurring in "pairs," and motivate the following definition: A space X is called a *strong quasi-complete space* if there exists a sequence $\langle\mathcal{U}_n\rangle$ of open covers of X such that if $\{x_i : i \geq n\} \cup \{x\} \subset U_n \in \mathcal{U}_n$, then the sequence $\langle x_n\rangle$ clusters to x. The sequence $\langle\mathcal{U}_n\rangle$ will be called a *strong quasi-complete sequence*. If the sequence $\langle\mathcal{U}_n\rangle$ satisfies condition (D) it will be called a *quasi-complete sequence*.

The basic implications among the concepts defined in (A), (B), (C) and (D) are given in the following diagram:

M-space	$\leftarrow$	metrizable
$\downarrow$		$\downarrow$
wM-space		
$\downarrow$		
wΔ-space	$\leftarrow$	developable
$\downarrow$		
quasi-complete	$\leftarrow$	strong quasi-complete

None of the implications are reversible; moreover, the space $[0,\Omega)$, where Ω is the first uncountable ordinal, is an M-space which is not strong quasi-complete. An example of a strong quasi-complete space which is not a developable space will be presented in Section 2 (see Example 2.4).

A natural question to ask at this point is the following: Under what conditions does a concept in the first column imply the corresponding concept in the second column? Several solutions are known and these will be discussed in Section 2; however, several questions remain open.

2. Strong Quasi-Complete Spaces

In this section we discuss the relationship of strong quasi-complete spaces to other classes of spaces, and determine when a quasi-complete space is strong quasi-complete.

In [8], Chaber proved the following important result.

Theorem 2.1 (Chaber [8]) *A* T_2*-space is metrizable if and only if it is an* M*-space with a* G_δ*-diagonal.*

In light of Chaber's result, the following problems become particularly interesting.

(1) Is every regular wM-space with a G_δ-diagonal metrizable?

(2) Is every regular wΔ-space with a G_δ-diagonal developable?

(3) Is every regular quasi-complete space with a G_δ-diagonal strong quasi complete?

In Theorem 2.2 we show that question (3) has a positive answer. This is particularly interesting since questions (1) and (2) remain open even if we assume collectionwise normality. Several partial solutions of questions (1) and (2) are known. For example, it follows from results in [17] that positive solutions are obtained if we assume θ-refinability or if we replace G_δ-diagonal by G_δ^*-diagonal or $\sigma^\#$-space (= α-space [17, Lemma 4.8]).

A space X is called a *p-space* if there exists a sequence $\langle \mathcal{G}_n \rangle$ of open covers of X satisfying: If $x \in X$ and $G_n \in \mathcal{G}_n$ such that $x \in G_n$, then

(a) $\cap_{n=1}^{\infty} \overline{G}_n$ is compact;

(b) if $x_n \in \cap_{i=1}^{n} \overline{G}_i$, then the sequence $\langle x_n \rangle$ clusters.

The class of p-spaces was introduced by Arhangel'skiĭ [2]; however, the above definition is the characterization obtained by Burke [5] (complete regularity is not assumed in our definition). Recall that for regular spaces, condition (b) is a characterization of quasi-complete spaces [15].

According to Ceder [7], a space X has a G_δ-diagonal if and only if there is a sequence $<\mathcal{G}_n>$ of open covers of X such that $\bigcap_{n=1}^{\infty} St(x, G_n) = \{x\}$ for every $x \in X$. The sequence $<\mathcal{G}_n>$ will be called a *G_δ-diagonal sequence.*

Theorem 2.2. For a regular space X, *the following are equivalent:*

(a) X *is a strong quasi-complete space.*

(b) X *is a* p-*space with a* G_δ-*diagonal.*

(c) X *is a quasi-complete space with a* G_δ-*diagonal.*

Proof. (a) → (b): Since every regular quasi-complete space with a G_δ-diagonal is a p-space [15, Theorem 3.6], it suffices to show that a strong quasi-complete space has a G_δ-diagonal. Let $<\mathcal{U}_n>$ be a strong quasi-complete sequence for X. Suppose x, y $\in$ X with $x \neq y$. If $y \in \bigcap_{n=1}^{\infty} St(x, \mathcal{U}_n)$, then there exists a $U_n \in \mathcal{U}_n$ such that $\{x,y\} \subset U_n$ for every $n \in N$. It follows that the constant sequence <y> must cluster to x. Since this is impossible, $<\mathcal{U}_n>$ is a G_δ-diagonal sequence for X.

The fact that (b) → (c) is a consequence of [15, Lemma 3.3].

(c) → (a): Let $<\mathcal{U}_n>$ be a quasi-complete sequence for X, and let $<\mathcal{G}_n>$ be a G_δ-diagonal sequence for X. For each $n \in N$, let $\mathcal{V}_n = \mathcal{U}_n \wedge \mathcal{G}_n = \{U \cap G: U \in \mathcal{U}_n, G \in \mathcal{G}_n\}$. By regularity, there is an open cover $\mathcal{W}_1$ of X such that $\{\overline{W}: W \in \mathcal{W}_1\} < \mathcal{V}_1$ and, for each $n \geq 2$, an open cover $\mathcal{W}_n$ of X such that $\{\overline{W}: W \in \mathcal{W}_n\} < \mathcal{V}_n \wedge \mathcal{W}_{n-1}$. We note that $<\mathcal{W}_n>$ is both a quasi-complete sequence and a G_δ-diagonal sequence for X. If $\{x_i: i \geq n\} \cup \{x\} \subset W_n \in \mathcal{W}_n$, then the sequence $\langle x_n \rangle$ has a cluster point y. Since $x_k \in \overline{W}_n$ for every $k \geq n$, $y \notin X - \overline{W}_n$ for any $n \in N$. Hence $y \in \bigcap_{n=1}^{\infty} \overline{W}_n \subset \bigcap_{n=1}^{\infty} St(x, \mathcal{V}_n) = \{x\}$. Thus y = x and $\langle \mathcal{W}_n \rangle$ is a strong quasi-complete sequence.

A *base of countable order* [22] for a space X is a base $\mathcal{B}$ such that if $\mathcal{C}$ is a perfectly decreasing subcollection of $\mathcal{B}$ (i.e. $\mathcal{C}$ contains a proper subset of each of its members) and $x \in \bigcap\{C: C \in \mathcal{C}\}$, then $\mathcal{C}$ is a local base at x.

Theorem 2.3. Every regular strong quasi-complete space X *has a base of countable order.*

Proof. Let $\langle \mathcal{U}_n \rangle$ be a strong quasi-complete sequence for X. For each $n \in N$, let $\mathcal{V}_n = \{V \text{ open in } X: \overline{V} \subset U \in \mathcal{U}_n\}$ and note that $\mathcal{V}_n$ is a base for X. Suppose $x \in V_n \in \mathcal{V}_n$ and $V_{n+1} \subset V_n$. Since the proof of (a) → (b) in Theorem 2.2 shows that $\langle \mathcal{U}_n \rangle$ is a G_δ-diagonal sequence, it follows easily that $\bigcap_{n=1}^{\infty} \overline{V}_n = \{x\}$. Let W be any open set such that $x \in W$. If $x_n \in V_n - W$, then $\{x_i: i \geq n\} \cup \{x\} \subset V_n$ because $V_i \subset V_n$ if $i > n$. Hence the sequence $\langle x_n \rangle$ clusters to x which is a contradiction. Thus $\langle V_n \rangle$ is a local base at x, and so X has a base of countable order [22, Theorem 2].

It follows from results in [22], that every regular, θ-refinable, strong quasi-complete space is developable, and that every paracompact, strong quasi-complete T_2-space is metrizable.

In order to dispel any thought on the part of the reader that the concept of strong quasi-complete might be equivalent to either developable or base of countable order, we site the following examples.

Example 2.4. A collectionwise normal, strong quasi-complete space which is not a developable space.

The space Λ constructed by van Douwen in [11] is such a space. Since Λ is locally compact and submetrizable, Λ is a strong quasi-complete space. It follows easily from the fact that Λ is normal and ω_1-compact, that Λ is collectionwise normal. Since Λ is not metrizable, Λ is not developable [3, Theorem 10].

The space $[0,\Omega)$ shows that a space with a base of countable order need not be strong quasi-complete.

Several examples exist which show that strong quasi-complete spaces do not possess some of the well-known properties possessed by developable spaces. Recall that every developable T_1-space is θ-refinable [22]; however, the space Σ constructed by van Douwen in [11] is a strong quasi-complete space which is not even countably θ-refinable. Actually, van Douwen shows that Σ is not countably metacompact; however, a countably θ-refinable space is countably metacompact [14]. The space Γ of van Douwen and Wicke [12] is a strong quasi-complete space which is not even countably orthocompact (note that Σ is orthocompact).

As was shown earlier, every strong quasi-complete space has a G_δ-diagonal; however, Burke's Example [6] shows that a completely regular quasi-complete space need not have a G_δ^*-diagonal. On the other hand, a regular developable space is easily seen to have a G_δ^*-diagonal.

In the discussion following questions (1), (2) and (3) we noted that replacing G_δ-diagonal by $\sigma^\#$-space gives a positive answer to questions (1) and (2). However, we do not know the answer to the following:

(4) Is every regular quasi-complete, $\sigma^\#$-space a strong quasi-complete space?

3. Properties of Strong Quasi-Complete Spaces

In this section we discuss some of the topological properties of strong quasi-complete spaces. Before doing this, however, let us give some alternate characterizations of strong quasi-complete spaces. The proofs will be left to the reader.

Theorem 3.1. For a space X, *the following are equivalent:*

(i) X *is a strong quasi-complete space.*

(ii) *There exists a sequence* $\langle \mathcal{U}_n \rangle$ *of open covers of* X *such that if* $x \in U_n \in \mathcal{U}_n$, *then* $\{U_n: n \in N\}$ *is a local subbase at* x.

(iii) *There exists a sequence* $\langle \mathcal{U}_n \rangle$ *of open covers of* X *such that if* $\{x_i: i \geq n\} \subset \bigcup\{U: U \in \mathcal{U}_n'\}$ *and* $x \in \bigcap\{U: U \in \mathcal{U}_n'\}$ *for some finite subset* $\mathcal{U}_n' \subset \mathcal{U}_n$, *then* $\langle x_n \rangle$ *clusters to* x.

The equivalence of (i) and (ii) is easy and is noted in [13]. That (i) and (iii) are equivalent follows exactly as in [15, Lemma 3.4]. It is interesting to note that if we allow the finite collection $\mathcal{U}_n'$ in (iii) to be countable, we actually obtain a characterization of developable spaces.

Strong quasi-complete spaces exhibit much better behavior than quasi-complete spaces with respect to subspaces and products. It is known that quasi-complete spaces are not hereditary [15] and not countably productive [16].

Theorem 3.2. (a) *Every strong quasi-complete space is hereditarily strong quasi-complete.*

(b) *If* $\langle X_n \rangle$ *is a sequence of strong quasi-complete spaces, then* $X = \Pi_{n=1}^{\infty} X_n$ *is a strong quasi-complete space.*

Proof. The result in (a) follows easily from the characterization of strong quasi-complete spaces given in Theorem 3.1 (ii). That (b) holds follows from [16, Theorem 3.1].

In [6], Burke shows that the perfect image of a locally compact T_2-space with a G_δ-diagonal (hence a strong quasi-complete space) need not have a G_δ-diagonal. It follows that strong quasi-complete spaces need not be preserved by perfect maps. The space Y_2 of Chaber [9] shows that the open compact image of completely regular, metacompact, complete Moore space (hence a strong quasi-complete space) need not be quasi-complete nor have a G_δ-diagonal. In [21, Example 3.7], Tanaka constructs a regular, paracompact space with a G_δ-diagonal which is not metrizable, but which is the open finite-to-one preimage of a compact metric space. It follows that Tanaka's space is not strong quasi-complete. The space $[0,\Omega)$, where Ω is the first uncountable ordinal, is an M-space and thus the quasi-perfect preimage of a metrizable space [20, Theorem 6.1]. However, $[0,\Omega)$ does not have a G_δ-diagonal and is thus not strong quasi-complete.

Summarizing these results, we have:

(1) Strong quasi-complete spaces need not be preserved by perfect maps or open compact maps.

(2) The preimage of a strong quasi-complete space under an open finite-to-one map or a quasi-perfect map need not be a strong quasi-complete space.

References

1. P. S. Alexandroff and P. Urysohn, *Une condition nessaire et suffisante pour qu'une class* (L) *soit une class* (D), C. R. Acad. Sci. Paris 177 (1923), 1274-1277.
2. A. V. Arhangel'skiĭ, *On a class of spaces containing all metric spaces and all locally bicompact spaces*, Soviet Math. Dokl. 4 (1963), 1051-1055.
3. R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. 3 (1951), 175-186.
4. C. J. R. Borges, *On metrizability of topological spaces*, Canad. J. Math. 20 (1968), 795-804.
5. D. K. Burke, *On* p-*spaces and* wΔ-*spaces*, Pacific J. Math. 35 (1970), 285-296.
6. ________, *A nondevelopable locally compact Hausdorff space with a* G_δ-*diagonal*, Gen. Topology Appl. 2 (1972), 287-291.
7. J. G. Ceder, *Some generalizations of metric spaces*, Pacific J. Math. 11 (1961), 105-126.
8. J. Chaber, *Conditions which imply compactness in countably*

compact spaces, preprint.

9. ________, *Metacompactness and the class MOBI*, preprint.
10. G. D. Creede, *Concerning semi-stratifiable spaces*, Pacific J. Math. 32 (1970), 47-54.
11. E. K. van Douwen, *A technique for constructing honest locally compact submetrizable spaces*, preprint.
12. E. K. van Douwen and H. H. Wicke, *A real, weird topology on the reals*, preprint.
13. J. Gerlits, *On* G_δ p-*spaces*, Colloq. Math. Soc. Janos Bolyai 8 (1972), 341-346.
14. R. F. Gittings, *Some results on weak covering conditions*, Canad. J. Math. 26 (1974), 1152-1156.
15. ________, *Concerning quasi-complete spaces*, Gen. Topology Appl. 6 (1976), 73-89.
16. ________, *Products of generalized metric spaces*, preprint.
17. R. E. Hodel, *Moore spaces and* wΔ-*spaces*, Pacific J. Math. 38 (1971), 641-652.
18. T. Ishii, *On* wM-*spaces. I, II.* Proc. Japan Acad. 46 (1970), 5-15.
19. R. L. Moore, *A set of axioms for plane analysis situs*, Fund. Math. 25 (1935), 13-28.
20. K. Morita, *Products of normal spaces with metric spaces*, Math. Ann. 154 (1964), 365-382.
21. Y. Tanaka, *On open finite-to-one maps*, Bull. Tokyo Gakugei Univ., Ser. IV, 25 (1973), 1-13.
22. J. M. Worrell, Jr. and H. H. Wicke, *Characterizations of developable topological spaces*, Canad. J. Math. 17 (1965), 820-830.

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