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1. Introduction

In 1973 R. C. Briggs [5] introduced two properties, pre-paracompactness (ppc) and \aleph -preparacompactness (\aleph -ppc) and compared them with the properties of paracompactness and collectionwise normality in various q -spaces. The purpose of this paper is to show that most of the results obtained in [5] can be generalized, hence closing the somewhat large gap between these properties.

Definition 1.1 A T_2 space X is *preparacompact* (resp. \aleph -preparacompact) if each open cover of X has an open refinement $\mathcal{K} = \{H_\alpha: \alpha \in A\}$ such that, if $B \subseteq A$ is infinite (resp. uncountable) and if p_β and $q_\beta \in H_\beta$ for each $\beta \in B$ with $p_\alpha \neq p_\beta$ and $q_\alpha \neq q_\beta$ for $\alpha \neq \beta$, then the set $Q = \{q_\beta: \beta \in B\}$ has a limit point whenever $P = \{p_\beta: \beta \in B\}$ has a limit point. The notions of σ -ppe and σ - \aleph -ppe should be clear. Collections satisfying the above property will be called *ppe* (\aleph -ppe) *collections*.

Since neither of the above properties implies paracompactness, even in the presence of collectionwise normality, the special setting of q -spaces is chosen for their study.

Definition 1.2 A space X is called a *q -space* if each point $p \in X$ has a sequence of neighborhoods $\{N_i\}_{i=1}^\infty$ such that if $y_i \in N_i$ for each i with $y_i \neq y_j$ for $i \neq j$, then the set $\{y_i\}_{i=1}^\infty$ has a limit point.

In [5] Briggs obtained the following.

Theorem 1.3 Let X be a regular q -space. Then the following are equivalent:

- (1) X is paracompact.
- (2) X is \aleph -ppc and subparacompact.
- (3) X is \aleph -ppc and metacompact.

Since the notion of a θ -refinability of J. Worrell and H. Wicke [10] is a generalization of both subparacompactness and metacompactness, it is natural to ask whether the above result can be generalized accordingly. In §2 of this paper we actually obtain a much stronger result using the notion of irreducible spaces [6]. Theorems involving the properties of $\delta\theta$ -refinability [1] and weak $\overline{\delta\theta}$ -refinability [9] are obtained in §3, and in §4 it is shown that every σ - \aleph -ppc, normal q -space is collection-wise normal. Examples and open questions are also included in §4.

2. Irreducible q -spaces

Definition 2.1 An open cover \mathcal{G} of a topological space X is called *minimal* provided no proper subcollection of \mathcal{G} covers X . A space X is called *irreducible* if every open cover of X has a minimal open refinement.

The following lemmas are easy to verify and hence the proofs are omitted.

Lemma 2.2 Let $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ be an open cover of an irreducible space X . Then \mathcal{G} has a minimal refinement $\mathcal{H} = \{H_\beta : \beta \in B\}$ where $H_\beta \subseteq G_\beta$ for all $\beta \in B \subseteq A$.

Lemma 2.3 A cover $\mathcal{W} = \{W_\alpha : \alpha \in A\}$ is a minimal cover of X iff there exists a discrete collection of non-empty closed sets $\{F_\alpha : \alpha \in A\}$ such that $F_\alpha \subseteq W_\alpha$ for each $\alpha \in A$.

Theorem 2.4 Let X be a q -space and let $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ be a \aleph -ppc collection of open subsets of X . If there exists a discrete collection $\{D_\beta : \beta \in B\}$ of non-empty subsets of X such

that $D_\beta \subseteq G_\beta$ for each $\beta \in B \subseteq A$, then $\{G_\beta: \beta \in B\}$ is either countable or locally finite.

Proof: Suppose B is uncountable and $\{G_\beta: \beta \in B\}$ is not locally finite at $p \in X$. Since X is a q -space, there exists a countable subcollection $\{G_{\beta_i}\}_{i=1}^\infty$ of \mathcal{G} and a sequence of points $\{p_i\}_{i=1}^\infty$ such that

- (i) $p_i \in G_{\beta_i}$ for each i ,
- (ii) $p_i \neq p_j$ and $G_{\beta_i} \neq G_{\beta_j}$ for $i \neq j$,
- (iii) $\{p_i\}_{i=1}^\infty$ has a limit point in X .

Now let $q_\beta \in D_\beta$ for each $\beta \in B$ and define $p_\beta = q_\beta$ for all $\beta \notin \{\beta_i: i=1,2,\dots\}$. Then $P = \{p_\beta: \beta \in B\}$ has a limit point while $Q = \{q_\beta: \beta \in B\}$ does not. This contradicts the fact that \mathcal{G} is an \aleph -ppc collection. Hence $\{G_\beta: \beta \in B\}$ is locally finite.

Remark: If \aleph -ppc is replaced by ppc in the above theorem then $\{G_\beta: \beta \in B\}$ is locally finite in each case.

Theorem 2.5 Let X be a regular q -space. Then X is paracompact iff X is \aleph -ppc and irreducible.

Proof: The necessity is clear. Let X be \aleph -ppc and irreducible and let \mathcal{U} be any open cover of X . Then \mathcal{U} has an open \aleph -ppc refinement $\mathcal{G} = \{G_\alpha: \alpha \in A\}$. Since X is irreducible \mathcal{G} has an open refinement \mathcal{K} which covers X minimally. By Lemma 2.2 above we may assume that $\mathcal{K} = \{H_\beta: \beta \in B\}$ where $H_\beta \subseteq G_\beta$ for each $\beta \in B \subseteq A$. By Lemma 2.3 there exists a discrete collection of non-empty closed sets $\{D_\beta: \beta \in B\}$ such that $D_\beta \subseteq H_\beta$ for each $\beta \in B$. Therefore, $\{G_\beta: \beta \in B\}$ is a σ -locally finite open refinement of \mathcal{U} , and hence X is paracompact by Theorem 1 of [7].

Corollary 2.6 Let X be a q -space. Then X is paracompact iff X is ppc and irreducible.

Proof: The proof follows immediately from the remark after Theorem 2.4 above.

Corollary 2.7 Let X be a regular q -space. Then the following are equivalent:

- (1) X is paracompact.
- (2) X is \aleph -ppc and θ -refinable.
- (3) X is \aleph -ppc and weak $\overline{\theta}$ -refinable.

Proof: In [9] the author has shown that θ -refinable and weak $\overline{\theta}$ -refinable spaces are irreducible.

Remark: It should be noted at this point that the above results (assuming regularity) remain true when \aleph -ppc is replaced by σ - \aleph -ppc by Theorem 2.4.

3. $\delta\theta$ -refinable Spaces

In [1] Aull proved that \aleph_1 -compact $\delta\theta$ -refinable spaces are Lindelöf and in [8] the author obtained an analogous result for weak $\overline{\delta\theta}$ -refinable spaces.

Definition 3.1 A space X is called $\delta\theta$ -refinable if every open cover \mathcal{G} of X has a refinement $\mathcal{G} = \bigcup_{i=1}^{\infty} \mathcal{G}_i$ satisfying,

- (i) each \mathcal{G}_i is an open cover of X .
- (ii) for each $x \in X$ there exists an integer $n(x)$ such that $\text{ord}(x, \mathcal{G}_{n(x)}) \leq \aleph_0$.

Definition 3.2 A space X is called weak $\overline{\delta\theta}$ -refinable if every open cover of X has a refinement $\mathcal{G} = \bigcup_{i=1}^{\infty} \mathcal{G}_i$ satisfying,

- (i) each \mathcal{G}_i is a collection of open subsets of X .
- (ii) for each $x \in X$ there exists an integer $n(x)$ such that $0 < \text{ord}(x, \mathcal{G}_{n(x)}) \leq \aleph_0$.
- (iii) $\{G_i^* = \bigcup \{G : G \in \mathcal{G}_i\}\}_{i=1}^{\infty}$ is point finite.

Even though $\delta\theta$ -refinable spaces need not be irreducible it is natural to ask whether similar results to those in §2 can be obtained since such spaces are generalizations of θ -refinable spaces. Here we provide such results using the notion of maximal

distinguished sets, due to Aull [1].

Let \mathcal{U} be an open cover of a topological space X .

Definition 3.3 A set M is distinguished with respect to \mathcal{U} if for each pair $x, y \in M$ with $x \neq y$, then $x \in U \in \mathcal{U} \Rightarrow y \notin U$.

Lemma 3.4 For every subset M of a space X and every open (in X) cover \mathcal{U} of M , there exists a maximal distinguished set with respect to \mathcal{U} which is discrete in $\bigcup \{U: U \in \mathcal{U}\}$.

Theorem 3.5 Let X be a regular q -space. Then X is paracompact iff X is \aleph -ppc and $\delta\theta$ -refinable.

Proof: Let X be \aleph -ppc and $\delta\theta$ -refinable and let \mathcal{U} be an open cover of X . Then \mathcal{U} has an \aleph -ppc refinement $\mathcal{G} = \{G_\alpha: \alpha \in A\}$. Since X is $\delta\theta$ -refinable, \mathcal{G} has a refinement $\bigcup_{i=1}^\infty \mathcal{W}_i$ satisfying,

- (i) each $\mathcal{W}_i = \{W(\alpha, i): \alpha \in A\}$ is an open cover of X ,
- (ii) for each $x \in X$, there exists an integer $n(x)$ such that $\text{ord}(x, \mathcal{G}_{n(x)}) \leq \aleph_0$.

As before we may assume $W(\alpha, i) \subseteq G_\alpha$ for each $\alpha \in A$ and each i . Now let $H_n = \{x: \text{ord}(x, \mathcal{G}_n) \leq \aleph_0\}$ so that $X = \bigcup_{n=1}^\infty H_n$. Let M_n be a maximal distinguished set of H_n with respect to \mathcal{G}_n for each n . By Lemma 3.4 the collection of singletons of points of each M_n is a discrete collection in X . By Theorem 2.4 above H_n is covered by a σ -locally finite subcollection of \mathcal{W}_n for each n . Therefore \mathcal{U} has a σ -locally finite open refinement, and hence X is paracompact.

The analogous result for weak $\overline{\delta\theta}$ -refinable spaces is also true. The proof is a modification of the one above and hence is omitted.

Theorem 3.6 Let X be a regular q -space. Then X is paracompact iff X is \aleph -ppc and weak $\overline{\delta\theta}$ -refinable.

4. Normal- q -spaces

In [5] Briggs obtained the following result using a somewhat involved argument. We now generalize this result using a theorem of Zenor [11].

Theorem 4.1 (Briggs) Let X be a normal q -space. If X is \aleph -ppc, then X is collectionwise normal.

Theorem 4.2 (Zenor) A space X is collectionwise normal iff for each discrete collection $\{F_\alpha: \alpha \in A\}$ of closed sets, there exists a sequence of collections $\{V(\alpha, i): \alpha \in A\}_{i=1}^\infty$ of open subsets of X satisfying,

- (i) $\{V(\alpha, i)\}_{i=1}^\infty$ covers F_α for each $\alpha \in A$,
- (ii) $F_\alpha \cap [\bigcup_{\beta \neq \alpha} V(\beta, i)]^- = \emptyset$ for each $\alpha \in A$ and each i .

Theorem 4.3 Let X be a normal q -space. If X is σ - \aleph -ppc, then X is collectionwise normal.

Proof: Let $\{F_\alpha: \alpha \in A\}$ be an uncountable discrete collection of closed subsets of X . Since X is normal there exists for each $\alpha \in A$ an open set G_α containing F_α such that $\overline{G_\alpha} \cap [\bigcup_{\beta \neq \alpha} F_\beta] = \emptyset$. We may assume that $0 \notin A$. Then let $G_0 = X - [\bigcup_{\alpha \in A} F_\alpha]$, and $\mathcal{G} = \{G_\alpha: \alpha \in A\} \cup \{G_0\}$. Since X is σ - \aleph -ppc, \mathcal{G} has a refinement $\bigcup_{i=1}^\infty \mathcal{H}_i$ where $\mathcal{H}_i = \{H(\alpha, i): \alpha \in A\}$ has the \aleph -ppc property and $H(\alpha, i) \subseteq G_\alpha$ for each $\alpha \in A$ and each i . Let $\mathcal{H}_i^* = \{H(\alpha, i): H(\alpha, i) \cap F_\alpha \neq \emptyset\}$ for each i . Then by Theorem 2.4, each \mathcal{H}_i^* is either countable or locally finite so that $\{H(\alpha, i): \alpha \in A\}_{i=1}^\infty$ satisfies the conditions of Theorem 4.2 above. Therefore X is collectionwise normal.

Briggs [5] used several examples to demonstrate the necessity of a special setting (q -spaces) in order to study the relationships between precompact spaces and other more common generalizations of paracompactness. These examples are summarized here for the benefit of the reader. For more details see [5].

Example I: A countably compact, first countable, normal q -space which is ppc and collectionwise normal but not paracompact.

Example II: A first countable, collectionwise normal q -space which is not \aleph - ppc .

Example III: A normal, metacompact, ppc space which is not collectionwise normal.

Example IV: A regular, locally countably compact q -space which is \aleph - ppc and σ - ppc but not ppc .

Example V: A regular, countably compact, q -space which is ppc but not normal.

Example VI: A metacompact, first countable, Lindelöf q -space which is \aleph - ppc but not regular.

Several interesting open questions remain:

- (1) Is every regular, first countable, ppc space normal?
- (2) Is Theorem 3.5 true for weak θ -refinable spaces?
- (3) In what setting, other than q -spaces, are the above results true?
- (4) When are ppc spaces expandable?
- (5) When are \aleph - ppc spaces countably paracompact?

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