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by

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1. Introduction

Suppose X and Y are topological spaces and $f: X \to Y$ is a perfect mapping (i.e., f is closed, continuous, onto, and $f^{-1}(y)$ is compact for every $y \in Y$). There are several theorems in the literature which indicate that certain base axioms are preserved under such a map. Two important results of this type were given by Worrell and Filippov:

Theorem 1.1 [Wo]: *If* X *is developable and* $f: X \to Y$ *is a perfect mapping then* Y *is developable.*

Theorem 1.2 [Fi]: *If a* T_1 *space* X *has a point-countable base and* $f: X \to Y$ *is a perfect mapping then* Y *has a point-countable base.*

An alternate approach to the proof of Theorem 1.2 was given by the following characterization of spaces with a point-countable base.

Theorem 1.3 [BM]: *The following properties of a space* Y *are equivalent:*

(a) Y *has a point-countable base.*

(b) Y *has a point-countable cover* $\mathcal{P}$ *such that if* $y \in W$ *with* W *open in* Y *there is a finite subcollection* $\mathcal{F}$ *of* $\mathcal{P}$ *such that* $y \in (\bigcup \mathcal{F})^\circ \subset (\bigcup \mathcal{F}) \subset W$ *and* $y \in \bigcap \mathcal{F}$.

If a T_1 space X has a point-countable base $\mathcal{B}$ and $f: X \to Y$ is a perfect map then the compact set $f^{-1}(y)$ intersects only countably many members of $\mathcal{B}$[Mi] for every $y \in Y$. If $\mathcal{P} = f(\mathcal{B}) = \{f(B): B \in \mathcal{B}\}$, it follows easily that condition (b) of

Theorem 1.3 is satisfied, so Theorem 1.2 is an immediate corollary to Theorem 1.3.

Techniques similar to those used in the proof of Theorem 1.3 have been used by the author to partially answer the question of whether the perfect image of a quasi-developable space is quasi-developable. In the course of this investigation a characterization of developable spaces was obtained which gives Worrell's result (Theorem 1.1) as a corollary. This characterization is given below, as well as the partial results for quasi-developable spaces. We conclude the paper by including a proof of the result that the perfect image of a space with a σ-point-finite base has a σ-point-finite base [Fi], and by giving an example to show that the corresponding result does not hold for spaces with a σ-disjoint base.

2. A Characterization of Developable Spaces

In order to state and prove the main theorem it will be necessary to define the idea of a pair-network and develop some companion notation.

A collection $\mathcal{L} = \{(Q_\alpha, R_\alpha): \alpha \in \Lambda\}$ of pairs of subsets of a space X is called a *pair-network* for X if whenever $x \in W$, with W open in X, there is some $P = (Q_\alpha, R_\alpha) \in \mathcal{L}$ such that $x \in Q_\alpha \subset R_\alpha \subset W$. The notion of a pair-network is not new and was used in [Ko] to define a class of spaces which coincides with the class of semi-stratifiable spaces.

If $\mathcal{L}$ is a pair-network for X and $P \in \mathcal{L}$ we let P' denote the first element in the pair P and let P" denote the second element. If $\mathcal{R} \subset \mathcal{L}$ let $\mathcal{R}' = \{P' : P \in \mathcal{R}\}$ and $\mathcal{R}'' = \{P'' : P \in \mathcal{R}\}$. If $x \in X$ and $\mathcal{R} \subset \mathcal{L}$ let

$St(x, \mathcal{R}) = \bigcup\{P'' : P \in \mathcal{R}, x \in P'\}$, and if $A \subset X$ then

$St(A, \mathcal{R}) = \bigcup\{P'' : P \in \mathcal{R}, A \cap P' \neq \emptyset\}$.

When $\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_n$ are subcollections of $\mathcal{L}$ we define

$\mathcal{R}_1 \wedge \mathcal{R}_2 \wedge \cdots \wedge \mathcal{R}_n$ to be the collection of all pairs of the form $(P_1' \cap P_2' \cap \cdots \cap P_n', P_1'' \cap P_2'' \cap \cdots \cap P_n'')$ such that $P_i \in \mathcal{R}_i$, $i = 1,2,\cdots,n$.

Recall that a *quasi-development* [Be] for a space Y is a sequence $\{\mathcal{G}_n\}_1^\infty$ of collections of open subsets of Y such that if $x \in U \subset Y$ where U is open in Y, there is some n such that $x \in St(x, \mathcal{G}_n) \subset U$. Y is developable if and only if Y is quasi-developable and every open subset of Y is an F_σ-set [Be].

Theorem 2.1. The following properties of a space Y *are equivalent:*

(a) Y *is developable.*

(b) Y *has a pair-network* $\mathcal{L} = \bigcup_{n=1}^\infty \mathcal{L}_n$ *satisfying:*

(i) Each $\mathcal{L}_n'$ *is a locally finite collection of closed sets and* $\mathcal{L}_n''$ *is a collection of open sets.*

(ii) Whenever $C \subset U \subset Y$ *where* C *is compact and* U *is open, there is some* $n \in N$ *such that* $C \subset St(C, \mathcal{L}_n) \subset U$.

(c) Y *has a pair-network* $\mathcal{R} = \bigcup_{n=1}^\infty \mathcal{R}_n$ *satisfying:*

(i) Each $\mathcal{R}_n'$ *is a locally finite collection of closed sets.*

(ii) Whenever $x \in U \subset Y$ *with* U *open, there is some* $n \in N$ *such that* $x \in (St(x, \mathcal{R}_n))^\circ \subset U$.

Proof: (a) $\rightarrow$ (b). Let $\{\mathcal{G}_n\}_1^\infty$ be a development for Y where we may assume $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$. Since Y is subparacompact [Bu] each $\mathcal{G}_n$ has a closed refinement $\mathcal{F}_n = \bigcup_{k=1}^\infty \mathcal{F}(n,k)$, where each $\mathcal{F}(n,k)$ is discrete. If $\mathcal{G}_n = \{G_\alpha : \alpha \in \Lambda_n\}$, we may assume each $\mathcal{F}(n,k)$ can be expressed as $\mathcal{F}(n,k) = \{F(k,\alpha) : \alpha \in \Lambda_n\}$ where $F(k,\alpha) \subset G_\alpha$ for every $\alpha \in \Lambda_n$. Let $\mathcal{P}(n,k) = \{(F(k,\alpha), G_\alpha) : \alpha \in \Lambda_n\}$; then $\bigcup\{\mathcal{P}(n,k) : n,k \in N\}$ is a pair-network for X. For any finite sequence $k_1, k_2, \cdots, k_n$ of positive integers, define

$$\mathcal{Q}(k_1,k_2,\cdots,k_n) = \mathcal{P}(1,k_1) \wedge \mathcal{P}(2,k_2) \wedge \cdots \wedge \mathcal{P}(n,k_n).$$

Now suppose $C \subset U \subset Y$ where C is compact and U is open. Let

$x \in C$. Choose a sequence $\{k_i\}_{i=1}^{\infty}$ of positive integers such that x is in some element of $\mathcal{F}(i,k_i)$ for every $i \in N$. For each n, let

$$A_n = \bigcup\{Q': Q \in \mathcal{Q}(k_1,k_2,\cdots,k_n),\ Q' \cap C \neq \emptyset,\ Q'' \not\subset U\}.$$

Clearly $\{A_n\}_1^{\infty}$ is a decreasing sequence of closed sets and if each A_n is nonempty there must be some $z \in (\bigcap_{n=1}^{\infty} A_n) \cap C$. Let $m \in N$ such that $St(z, \mathcal{G}_m) \subset U$; it follows that

$$St(z, \mathcal{Q}(k_1,k_2,\cdots,k_m)) \subset St(x, \mathcal{G}_m) \subset U$$

and this will contradict the definition of A_m. Thus $A_n = \emptyset$ for some n and this implies $x \in St(C, \mathcal{Q}(k_1,\cdots,k_n)) \subset U$. Now let $\mathcal{L} = \bigcup\{\mathcal{Q}(k_1,\cdots,k_n): k_1,k_2,\cdots,k_n$ is a finite sequence of positive integers$\}$. Consider all collections obtained by taking unions of a finite number of elements of $\{\mathcal{Q}(k_1,\cdots,k_n): k_1,\cdots,k_n$ is a finite sequence of positive integers$\}$. These collections can be enumerated as $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3,\cdots$ and $\mathcal{L}$ can be expressed as $\mathcal{L} = \bigcup_{n=1}^{\infty} \mathcal{L}_n$ with $\mathcal{L}$ satisfying the conditions given in (b).

(b) $\rightarrow$ (c). Trivial.

(c) $\rightarrow$ (a). Let $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{R}_n$ be a pair-network as given in (c). For every $n,k \in N$ let

$$\Phi_{n,k} = \{\mathcal{F} \subset \mathcal{R}_n': |\mathcal{F}| = k\}.$$

and let

$$G(\mathcal{F}) = (\bigcup\{R'': R \in \mathcal{R}_n \text{ and } R' \in \mathcal{F}\})^{\circ} - \bigcup(\mathcal{R}_n' - \mathcal{F})$$

for every $\mathcal{F} \in \Phi_{n,k}$. Define $\mathcal{G}_{n,k} = \{G(\mathcal{F}): \mathcal{F} \in \Phi_{n,k}\}$. Since $\mathcal{R}'$ is a σ-locally finite network of closed sets it is clear that open subsets of Y are F_σ sets and it suffices to show that $\{\mathcal{G}_{n,k}: n,k \in N\}$ is a quasi-development for Y. To show this let $x \in U$ where U is open; by assumption there is some $m \in N$ such that $x \in (St(x,\mathcal{R}_m))^{\circ} \subset U$. Let $\mathcal{F} = \{R': R \in \mathcal{R}_m,\ x \in R'\}$; then $|\mathcal{F}| = k > 0$ for some $k \in N$ so $\mathcal{F} \in \Phi_{m,k}$. Clearly $x \in G(\mathcal{F}) \subset (St(x,\mathcal{R}_m))^{\circ} \subset U$. If $\mathcal{E} \in \Phi_{m,k}$ such that $\mathcal{E} \neq \mathcal{F}$ then $x \in \bigcap(\mathcal{R}_m' - \mathcal{E})$ and $x \notin G(\mathcal{E})$. This says that $G(\mathcal{F})$ is the only element of $\mathcal{G}_{m,k}$ which contains x. Thus $x \in G(\mathcal{F}) = St(x, \mathcal{G}_{m,k}) \subset U$.

That completes the proof of the theorem.

To see that Theorem 1.1 follows as a corollary to the preceding theorem suppose X is developable and $f: X \to Y$ is a perfect mapping. Let $\mathcal{P}$ be a pair-network for X satisfying the condition as in (b) of Theorem 2.1. If $\mathcal{R} = \{(f(P'), f(P'')): P \in \mathcal{P}\}$ it is easily verified that $\mathcal{R}$ is a pair-network for Y satisfying condition (c) of Theorem 2.1, so Y is developable.

3. Quasi-developable Spaces

We now turn to the question of when a quasi-development is preserved under a perfect map. In [BL] Bennett and Lutzer showed that if $\mathcal{U}$ is an open cover of a quasi-developable space X then $\mathcal{U}$ has a refinement $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ such that each $\mathcal{F}_n$ is discrete relative to $\bigcup\{F: F \in \mathcal{F}_n\}$. The next lemma exhibits a slightly stronger version of this covering property.

Lemma 3.1. Suppose $\{\mathcal{G}_n\}_1^{\infty}$ *is a quasi-development for a space* X. *If* $\mathcal{U}$ *is any collection of open subsets of* X *there is a refinement* $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ *of* $\mathcal{U}$ *such that each* $\mathcal{F}_n$ *is closed and discrete relative to* $(\bigcup \mathcal{U}) \cap (\bigcup \mathcal{G}_n)$.

Proof: Assume $\mathcal{U} = \{U_\alpha: \alpha \in \Lambda\}$ where Λ is well-ordered. For each $n \in N$, $\alpha \in \Lambda$, let

$$P_{n,\alpha} = \{x: x \in U_\alpha - (\bigcup_{\beta<\alpha} U_\beta),\ x \in St(x, \mathcal{G}_n) \subset U_\alpha\}$$

and let $F_{n,\alpha}$ be the closure of $P_{n,\alpha}$ relative to $(\bigcup \mathcal{U}) \cup (\cap \mathcal{G}_n)$. Let $x \in (\bigcup \mathcal{U}) \cap (\bigcup \mathcal{G}_n)$ and suppose α is the first element of Λ such that $x \in U_\alpha$. Clearly $U_\alpha \cap P_{n,\beta} = \emptyset$ if $\beta > \alpha$ so $U_\alpha \cap F_{n,\beta} = \emptyset$ if $\beta > \alpha$. If $\beta < \alpha$ and $St(x, \mathcal{G}_n) \cap F_{n,\beta} \neq \emptyset$, then there is some $z \in St(x, \mathcal{G}_n) \cap P_{n,\beta}$. This implies $x \in St(z, \mathcal{G}_n) \subset U_\beta$, a contradiction to our choice of U_α. Hence $St(x, \mathcal{G}_n) \cap F_{n,\beta} = \emptyset$ if $\beta < \alpha$. It follows that $U_\alpha \cap St(x, \mathcal{G}_n)$ is an open set about x which has empty intersection with $F_{n,\beta}$ for any $\beta \in \Lambda$, $\beta \neq \alpha$. This says that $\mathcal{F}_n = \{F_{n,\beta}: \beta \in \Lambda\}$ is discrete relative to $(\cup\mathcal{U}) \cap (\cup\mathcal{G}_n)$ and that $F_{n,\beta} \subset U_\beta$ for every $\beta \in \Lambda$. If $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ it is clear that $\cup\mathcal{F} = \cup\mathcal{U}$; that completes the proof of the lemma.

Theorem 3.2. Suppose $f: X \to Y$ *is a perfect mapping and* X *has a quasi-development* $\{\mathcal{G}_n\}_1^\infty$ *such that whenever* $x \in U \cap f^{-1}(y)$ *where* U *is open in* X *and* $y \in Y$ *then there is some* $m \in N$ *such that* $\mathcal{G}_m$ *covers* $f^{-1}(y)$ *and* $St(x,\mathcal{G}_m) \subset U$. *Then* Y *is quasi-developable.*

Proof: For each n let $H_n = \cup\mathcal{G}_n$ and suppose $\mathcal{G}_n = \{G_\alpha: \alpha \in \Lambda_n\}$. We may assume each H_n is saturated with respect to f. By Lemma 3.1, $\mathcal{G}_n$ has a refinement $\bigcup_{k=1}^{\infty} \mathcal{F}(n,k)$ where each $\mathcal{F}(n,k)$ is closed and discrete relative to $H_n \cap H_k$; we may also assume $\mathcal{F}(n,k)$ has the form $\mathcal{F}(n,k) = \{F_{k,\alpha}: \alpha \in \Lambda_n\}$ where each $F_{k,\alpha} \subset G_\alpha$. Let

$$\mathcal{P}(n,k) = \{(F_{k,\alpha}, G_\alpha \cap H_n \cap H_k): \alpha \in \Lambda_n\};$$

then $\cup\{\mathcal{P}(n,k): n,k \in N\}$ is a pair-network for X. For finite sequences $n_1,n_2,\cdots,n_r$ and $k_1,k_2,\cdots,k_r$ of positive integers, define

$$\mathcal{Q}(n_1,k_1,n_2,k_2,\cdots,n_r,k_r) = \mathcal{P}(n_1,k_1) \wedge \mathcal{P}(n_2,k_2) \wedge\cdots\wedge \mathcal{P}(n_r,k_r).$$

Now suppose $f^{-1}(y) \subset U \subset X$ where $y \in Y$ and U is open. Let $n_1,n_2,n_3,\cdots$ be a sequence of positive integers such that if $\mathcal{G}_m$ covers $f^{-1}(y)$ then $m = n_i$ for some i. Let $x \in f^{-1}(y)$. Choose a sequence $\{k_i\}_{i=1}^\infty$ of positive integers such that x is in some element of $\mathcal{F}(n_i,k_i)$ for every i. Using an argument similar to that used in the proof of (a) $\to$ (b) in Theorem 2.1 it follows that there is some $r \in N$ such that

$$x \in St(f^{-1}(y), \mathcal{Q}(n_1,k_1,n_2,k_2,\cdots,n_r,k_r)) \subset U.$$

Now let the family of all collections $\mathcal{Q}(n_1,k_1,\cdots,n_s,k_s)$, where $n_1,n_2,\cdots,n_s$ and $k_1,k_2,\cdots,k_s$ are finite sequences of positive integers, be enumerated as $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3,\cdots$. For a given $\mathcal{Q}_j$, say $\mathcal{Q}_j = \mathcal{Q}(n_1,k_1,\cdots,n_s,k_s)$, define $B_j = H_{n_1} \cap H_{k_1} \cap \cdots \cap H_{n_s} \cap H_{k_s}$. If $M \subset N$ is a finite set, define $B_M = \cap\{B_j: j \in M\}$ and let

$$\mathcal{R}_M = \{(Q' \cap B_M, Q'' \cap B_M): Q \in \mathcal{Q}_j, j \in M\}.$$

The family of all collections $\mathcal{R}_M$, where M is a finite subset of N, can be enumerated as $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \cdots$ and if $\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n$ then $\mathcal{P}$ satisfies: (1) Each $\mathcal{P}_n'$ is locally finite and closed relative to $\cup\mathcal{P}_n''$. (ii) If $f^{-1}(y) \subset U$ where U is open in X and $y \in Y$ there is some $m \in N$ such that $f^{-1}(y) \subset St(f^{-1}(y), \mathcal{P}_m) \subset U$.

For every $n,k \in N$ let

$$\Phi_{n,k} = \{\mathcal{F} \subset \mathcal{P}_n': |\mathcal{F}| = k\}$$

and let $G(\mathcal{F})$ be the saturated part (with respect of f) of

$$(\cup\{P'': P \in \mathcal{P}_n, P' \in \mathcal{F}\}) - \cup(\mathcal{P}_n' - \mathcal{F}).$$

Define $\mathcal{G}(n,k) = \{f(G(\mathcal{F})): \mathcal{F} \in \Phi_{n,k}\}$; we show $\{\mathcal{G}(n,k): n, k \in N\}$ is a quasi-development for Y. Let $y \in V \subset Y$ where V is open in Y. By (ii) above there is $m \in N$ such that

$$f^{-1}(y) \subset St(f^{-1}(y), \mathcal{P}_m) \subset f^{-1}(V).$$

Let $\mathcal{F} = \{P': P \in \mathcal{P}_m, f^{-1}(y) \cap P' \neq \emptyset\}$; then $|\mathcal{F}| = k > 0$ for some integer k, so $\mathcal{F} \in \Phi_{m,k}$. Clearly

$$f^{-1}(y) \subset G(\mathcal{F}) \subset St(f^{-1}(y), \mathcal{P}_m) \subset f^{-1}(V).$$

If $\mathcal{E} \in \Phi_{m,k}$ such that $\mathcal{E} \neq \mathcal{F}$ then $f^{-1}(y) \cap (\cup(\mathcal{P}_m' - \mathcal{E})) \neq \emptyset$ and $f^{-1}(y) \cap G(\mathcal{E}) = \emptyset$. This says that $f(G(\mathcal{F}))$ is the only element of $\mathcal{G}(m,k)$ that contains y. Thus

$$y \in f(G(\mathcal{F})) = St(y, \mathcal{G}(m,k)) \subset V.$$

That completes the proof of the theorem.

In general, a given quasi-development for a space X may not satisfy the hypothesis of Theorem 3.2, however the quasi-development can often be modified in order to obtain the desired condition. The next corollary gives one situation in which this is always the case. A *p-base* (point separating open cover) for a space X is a collection $\mathcal{B}$ of open sets such that whenever $x,y \in X$, $x \neq y$, there is some $B \in \mathcal{B}$ such that $x \in B$ and $y \notin B$.

Corollary 3.3. Let $f: X \to Y$ *be a perfect map. If* X *is* *quasi-developable and has a countable p-base then* Y *is quasi-developable.*

Proof: Suppose $\{\mathcal{G}_n\}_1^\infty$ is a quasi-development for X and $\mathcal{B} = \{B_n: n \in N\}$ is a countable p-base. For every $n,k \in N$ let $\mathcal{U}_{n,k} = \{G \cap B_k: G \in \mathcal{G}_n\}$. Let $\mathcal{H}_1, \mathcal{H}_2, \cdots$ be an enumeration of all collections obtained by taking unions of a finite number of elements of $\{\mathcal{U}_{n,k}: n,k \in N\}$. It is easily verified that if $x \in C \cap U$ where $C \subset X$ is compact and U is open there is some $m \in N$ such that $\mathcal{H}_m$ covers C and $St(x, \mathcal{H}_m) \subset U$. The corollary now follows from Theorem 3.2.

Corollary 3.4. Suppose X *is Hausdorff and* $f: X \to Y$ *is a perfect mapping such that* $f^{-1}(y)$ *is a singleton set for all but countably many* $y \in Y$. *If* X *is quasi-developable then so is* Y.

Proof: Let $E = \{y \in Y: |f^{-1}(y)| > 1\}$; then E is a countable set. For each $y \in E$ the compact subspace $f^{-1}(y)$ of X is quasi-developable and thus separable metrizable [Be]. There is a countable collection $\mathcal{F}(y)$ of closed subsets of $f^{-1}(y)$ such that whenever $x,z \in f^{-1}(y)$, $x \neq z$, then there is some $F \in \mathcal{F}(y)$ where $x \in F$ and $z \notin F$. Let $\mathcal{B} = \{X - F: F \in \mathcal{F}(y), y \in E\}$; then $\mathcal{B}$ is a countable open cover of X such that whenever $y \in Y$ and $x,z \in f^{-1}(y)$, $x \neq z$, then there is some $B \in \mathcal{B}$ such that $x \in B$, $z \notin B$. A construction similar to that used in Corollary 3.3 will now finish the proof.

Corollary 3.4 can also be proven directly without reference to Theorem 3.2. In this case one shows first that Y is first countable and then a quasi-development for Y is constructed by considering the points of E separately.

4. Spaces With a σ-point Finite Base

A base $\mathcal{B}$ for the topology of a space X is said to be σ-point-finite if $\mathcal{B}$ can be expressed as $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ where each $\mathcal{B}_n$

is point-finite. Filippov stated in [Fi] that the perfect image of a space with a σ-point-finite base has a σ-point-finite base, but he did not give an explicit proof. Some recent interest has been shown in seeing a proof of this result, and since a proof has not appeared in print we provide one here. This proof was obtained by the author several years ago while working on some related material with E. Michael. We begin with a lemma that may have some independent interest.

Lemma 4.1. If $\mathcal{L}$ *is a point-finite collection of subsets of* X, $A \subset X$, *and* $n \in N$, *then there are at most a finite number of minimal covers* $\mathcal{F}$ *of* A, *by elements of* $\mathcal{L}$, *such that* $|\mathcal{F}| = n$.

Proof: Suppose there is an infinite collection Φ of minimal covers (of A) consisting of subcollections from $\mathcal{L}$ of cardinality n. Pick a maximal collection $\mathcal{R} \subset \mathcal{L}$ such that $\mathcal{R} \subset \mathcal{F}$ for infinitely many members $\mathcal{F} \in \Phi$, and let $\Phi' = \{\mathcal{F} \in \Phi : \mathcal{R} \subset \mathcal{F}\}$. Clearly $0 \leq |\mathcal{R}| < n$, so $\mathcal{R}$ does not cover A and there is some $y \in A - (\cup \mathcal{R})$. Hence if $\mathcal{F} \in \Phi'$, there is some $F \in \mathcal{F} - \mathcal{R}$ such that $y \in F$. Since only finitely many elements of $\mathcal{L}$ contain y, there must be some $F_0 \in \mathcal{L}$ such that $y \in F_0 \in \mathcal{F} - \mathcal{R}$ for infinitely many members $\mathcal{F} \in \Phi'$. Then $\mathcal{R} \cup \{F_0\} \subset \mathcal{F}$ for infinitely many members $\mathcal{F}$ of Φ, which contradicts the maximal condition placed on $\mathcal{R}$.

Theorem 4.2 [Fi]. *If* X *has a* σ*-point-finite base and* $f: X \to Y$ *is a perfect mapping then* Y *has a* σ*-point-finite base.*

Proof: Suppose $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ is an open base for X where each $\mathcal{B}_n$ is point-finite. For each $n,k \in N$, let

$$\Phi_{n,k} = \{\mathcal{F} \subset \bigcup_{i=1}^{k} \mathcal{B}_i : |\mathcal{F}| = n\}.$$

For $\mathcal{F} \in \Phi_{n,k}$ let

$$\mathcal{M}_k(\mathcal{F}) = \{A \in \bigcup_{i=1}^{k} \mathcal{B}_i : \mathcal{F} \text{ is a minimal cover of } f^{-1}(f(A))\}.$$

and let $U_k(\mathcal{F}) = Y - f[X - \cup(\mathcal{M}_k(\mathcal{F}))]$. Define $\mathcal{U}_{n,k} = \{U_k(\mathcal{F}) : \mathcal{F} \in \Phi_{n,k}\}$ and $\mathcal{U} = \cup\{\mathcal{U}_{n,k} : n,k \in N\}$. To show that $\mathcal{U}$ is a base for Y, let $y \in W \subset Y$ where W is open in Y. Since

$f^{-1}(y)$ is compact, there is $n,k \in N$ and $\mathcal{F} \in \Phi_{n,k}$ such that $\mathcal{F}$ is a minimal cover of $f^{-1}(y)$ and $\cup \mathcal{F} \subset f^{-1}(W)$. Now, if $r \geq k$ and $A \in \bigcup_{i=1}^{r} \mathcal{B}_i$ such that $A \cap f^{-1}(y) \neq \emptyset$ and $f^{-1}(f(A)) \subset \cup \mathcal{F}$ then $A \in \mathcal{M}_r(\mathcal{F})$; thus r can be chosen large enough so that $f^{-1}(y) \subset \cup \mathcal{M}_r(\mathcal{F})$ and it follows that $y \in U_r(\mathcal{F}) \subset W$. To complete the proof of the theorem we show that each $\mathcal{U}_{n,k}$ is point-finite. Let $y \in Y$ and pick a fixed $x \in f^{-1}(y)$. If $y \in U_k(\mathcal{F}) \in \mathcal{U}_{n,k}$ (so $\mathcal{F} \in \Phi_{n,k}$) then $f^{-1}(y) \subset \cup \mathcal{M}_k(\mathcal{F})$ and $x \in A$ for some $A \in \mathcal{M}_k(\mathcal{F})$; since $x \in A$ for only finitely many $A \in \bigcup_{i=1}^{k} \mathcal{B}_i$ it suffices to prove that each A out of $\bigcup_{i=1}^{k} \mathcal{B}_i$ is in only finitely many $\mathcal{M}_k(\mathcal{F})$ for $\mathcal{F} \in \Phi_{n,k}$. But this follows from Lemma 4.1 and the definition of the $\mathcal{M}_k(\mathcal{F})$. That completes the proof of the theorem.

The following example, due to R. W. Heath and G. M. Reed, shows that a σ-disjoint base is not necessarily preserved under a perfect mapping.

Example 4.3. There is an example of a Moore space X with a σ-disjoint base and a perfect mapping $f: X \to Y$ where Y does not have a σ-disjoint base.

If R is the set of real numbers let $H = \{(x,y) \in R \times R: y > 0\}$, $X_0 = R \times \{0\}$, $X_1 = R \times \{-1\}$, and $X = H \cup X_0 \cup X_1$. Describe a local base for each point as follows: All points in H are isolated in X. If $a \in R$ and $n \in N$, let

$$U_n(a,0) = \{(a,0)\} \cup \{(x,y) \in H: x = y + a,\ y < 1/n\}$$

and

$$U_n(a,-1) = \{(a,-1)\} \cup \{(x,y) \in H: x = -y + a,\ y < 1/n\}.$$

Then $\{U_n(a,0)\}_{n=1}^{\infty}$ and $\{U_n(a,-1)\}_{n=1}^{\infty}$ are local bases at (a,0) and (a,-1) respectively. It is easily verified that this induces a topology on X making X a regular, developable space with a σ-disjoint base. Let Y be the quotient space obtained from X by identifying the points (a,0) and (a,-1) for each $a \in R$, and let F: X → Y be the corresponding quotient map. Then f is a perfect map, and Y does not have a σ-disjoint base. This last fact can be shown directly, or it can be noted that Y is homeomorphic to the space described in Example 1 of [He]. Heath has shown this example is a nonscreenable Moore space, and hence could not have a σ-disjoint base.

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