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Howard Cook and Laurie Gibson

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Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
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If X is a topological space, G is a collection of point sets covering X, and 0 is a point set, we denote by $st_1(0,G) = st(0,G)$ the union of all of the point sets of G which intersect 0; if x is a point, $st(x,G) = st(\{x\},G)$; and, if n is a positive integer, $st_{n+1}(0,G) = st_n(st(0,G),G)$. If j is a positive integer, we say that a family $\mathcal{H}$ of open covers of X has the j-link property provided that, for each two points p and q of X, there is an $H \in \mathcal{H}$ such that $q \notin st_j(p,H)$. And we say that the space X has the j-link property if there is a countable family $\mathcal{H}$ of open covers of X having the j-link property. We note that (1) a space has the 1-link property if, and only if, it has a G_δ-diagonal [2]; (2) a space with the 2-link property has a G_δ^*-diagonal [5]; (3) a space with the 3-link property has a regular G_δ-diagonal [10]; (4) every Moore space having the j-link property has a development having the j-link property; and (5) if a Moore space has the j-link property for every positive integer j, then it has a development having the j-link property for every j. F. G. Slaughter, Jr. has pointed out that every submetrizable space has the j-link property for every j. G. M. Reed has shown [8] that there is a metacompact, continuously semimetrizable (and, therefore, Moore) space having the 3-link property but not the 4-link property. Such a space may have any uncountable cardinality but no such space is separable.

In this paper we construct, for each $j > 1$, a Moore space having the j-link property but not the (j+1)-link property. Such spaces may be separable and locally connected, separable and locally compact, or metacompact and continuously semimetrizable. But no such space can be both separable and continuously semimetrizable, for such spaces are submetrizable [3]; and if $j > 2$, no such space is both locally connected and locally compact, for such spaces are metrizable [4]. Of course all such spaces must be uncountable and all of our spaces have cardinality c. In the absence of the continuum hypothesis we may reasonably ask the following two questions.

Question 1. Is there a separable Moore space of cardinality less than c which has the 3-link property but fails to have the j-link property for some j?

Question 2. Is there a Moore space of cardinality less than c which has the 4-link property but fails to have the j-link property for some j?

All of the spaces constructed here are variations of a construction due to F. B. Jones [6].

The Spaces D_j, C_j, and V_j

Let $j > 1$ be an integer. In E^3 let $P_1, P_2, \cdots, P_j$ be distinct open half planes such that, if $1 \leq i \leq j$, $\overline{P}_i \backslash P_i$ is the x-axis. Let the x-axis be the union of j mutually exclusive point sets $A_1, A_2, \cdots, A_j$ such that, if $1 \leq i \leq j$, every uncountable closed subset of the x-axis intersects A_i [7, p. 514]. Let

$$D_j = V_j = [\cup_{i \leq i \leq j} P_i] \cup [\cup_{2 \leq i \leq j} A_i] \cup [A_1 \times \{0,1\}].$$

If $2 \leq i \leq j$, $\varepsilon > 0$, and $a \in A_i$, let $R_\varepsilon(a)$ denote the point set comprising (1) the bounded component of $P_{i-1} \backslash J$, where J is the circle containing a of radius ε lying in $P_{i-1} \cup \{a\}$; (2) the bounded component of $P_i \backslash C$, where C is the circle containing a of radius ε and lying in $P_i \cup \{a\}$; and (3) $\{a\}$. If $a \in A_1$ and $\varepsilon > 0$, let $R_\varepsilon((a,0))$ denote the point set comprising $\{(a,0)\}$ and the bounded component of the complement in $P_1 \cup \{a\}$ of the circle containing a of radius ε lying in $P_1 \cup \{a\}$; and let $R_\varepsilon((a,1))$ denote the point set comprising $\{(a,1)\}$ and the bounded component of the complement in $P_j \cup \{a\}$ of the circle containing a of radius ε lying in $P_j \cup \{a\}$. If $x \in \bigcup_{1 \leq i \leq j} P_i$, let $R_\varepsilon(x)$ be the intersection of $\bigcup_{1 \leq i \leq j} P_i$ and some open subset of E^3 of diameter less than ε which does not intersect the x-axis. Let $\{R_\varepsilon(x) | x \in D_j, \varepsilon > 0\}$ be the basis for our topology on D_j. Then if, for each n, $G_n = \{R_\varepsilon(x) | x \in D_j; 0 < \varepsilon < \frac{1}{n}\}$ the sequence $G_1, G_2, \cdots$ is a development for the Moore space D_j having the j-link property; D_j is separable and locally connected.

If $2 \leq i \leq j$, $\varepsilon > 0$, and $a \in A_i$, let $N_\varepsilon(a)$ be the set comprising a together with all points of $P_{i-1} \cup P_i$ at a distance less than ε from a which lie on a line containing a and forming a 45° angle with the x-axis. If $a \in A_1$, and $\varepsilon > 0$, let $N_\varepsilon((a,0))$ be the set comprising (a,0) together with all points of P_1 at a distance less than ε from a and on a line containing a and forming a 45° angle with the x-axis. If $x \in \bigcup_{1 \leq i \leq j} P_i$ and $\varepsilon > 0$, let $N_\varepsilon(x) = \{x\}$. Let $\{N_\varepsilon(x) | x \in V_j, \varepsilon > 0\}$ be a basis for our topology on V_j. Then if, for each n, $G_n = \{N_\varepsilon(x) | x \in \mathcal{S}, 0 < \varepsilon < \frac{1}{n}\}$ the sequence $G_1, G_2, \cdots$ is a development for the metacompact Moore space V_j having the j-link property. It is easy to show that V_j is continuously semimetrizable.

We shall now describe the subspace C_j of D_j for $j > 1$. Let $K_1, K_2, \cdots$ be a sequence such that (1) K_1 is an interval of the x-axis of length 1, and (2) for each positive integer n, K_{n+1} is a collection of 2^n intervals of length $1/3^n$ such that each interval of K_{n+1} is a subset of some interval of K_n and contains an end point of that interval of K_n. Let $K = \bigcup_{n>o} X_n$ and let $\mathcal{C} = \bigcap_{n>o} K_n^*$.[1] For each interval $k \in K$ and each i, $1 \leq i \leq j$, let P_{ik} be the point of P_i on the perpendicular bisector in $\overline{P}_i$ of k at a distance from the x-axis equal to the length of k.

[1]If K is a set collection, then K* denotes the union of the members of K.

Let $C_j = \{P_{ik} | k \in K, 1 \leq i \leq j\} \cup [\mathcal{C} \cap (\cup_{2 \leq i \leq j} A_i)] \cup [\mathcal{C} \cap A_1] \times \{0,1\}$. Then C_j is a separable and locally compact subspace of D_j and has the j-link property.

Theorem. If $j > 1$, *neither* D_j, C_j *nor* V_j *has the* (j+1)-link *property.*

Proof. Suppose that $\mathcal{H}$ is a countable family of open covers of C_j having the (j+1)-link property. Then there exist an element H of $\mathcal{H}$, a positive number ε_1, and an uncountable subset X_1 of $A_1 \cap \mathcal{C}$ such that, if $a \in X_1$, then $R_{\varepsilon_1}((a,0))$ is a subset of some element of H, $R_{\varepsilon_1}((a,1))$ is a subset of some element of H, and $(a,1) \notin st_{j+1}((a,0),H)$. There exist finite sequences $X_2, X_3, \cdots, X_j$ and $\varepsilon_2, \varepsilon_3, \cdots, \varepsilon_j$ such that, if $2 \leq i \leq j$, $0 < \varepsilon_i < \varepsilon_{i+1}$; $X_i \subset [\overline{X}_{i-1} \cap A_i]$ and is uncountable (where, if $X = C_j$, $\overline{X}$ denotes the closure in E^3 of X); and, if $a \in X_1$ then $R_{\varepsilon_i}(a)$ is a subset of some element of H. Let $x_1, x_2, \cdots, x_j$ be a finite sequence of points of $\overline{X}_j$ such that, for each i, $1 \leq i \leq j$, $x_i \in X_i$ and the distance in E^3 from x_i to x_1 is less than $\varepsilon_j/6$. Then $C_j \cap R_{\varepsilon_j}((x_1,0)) \cap R_{\varepsilon_j}(x_2) \neq \emptyset$; $C_j \cap R_{\varepsilon_j}((x_1,1)) \cap R_{\varepsilon_j}(x_j) \neq \emptyset$; and if $3 \leq i \leq j$, $C_j \quad R_{\varepsilon_j}(x_{i-1}) \cap R_{\varepsilon_j}(x_i) \neq \emptyset$. Then $(x_1,1) \in st_{j+1}((x_1,0),H)$, a contradiction. Thus C_j does not have the (j+1)-link property and, hence, neither does D_j.

That V_j does not have the (j+1)-link property follows from an entirely similar argument using $N_\varepsilon(x)$ instead of $R_\varepsilon(x)$ in every possible instance.

Two More Examples

A space X is said to be submetrizable if there is a continuous 1-1 mapping of X onto a metric space. Clearly, if p and q are two points of a submetrizable space X then there is a continuous real-valued function f on X such that $f(p) \neq f(q)$. We construct here a Moore space having the j-link property for every j which is not submetrizable and which has a submetrizable subspace which is not completely regular.

Let Z denote the set of all integers.

For each $i \in Z$, let P_i be an open half plane in E^3 each point of which has positive third coordinate such that (1) P_i separates P_{i-1} from P_{i+1} in $\{(x,y,z) \in E^3 | z > 0\}$; (2) $\overline{P}_i \backslash P_i$ is the x-axis; (3) $P_i \neq P_j$ if $i \neq j$. Let w be a point of the x-axis and let the x-axis minus $\{w\}$ be the sum of countably many mutually exclusive point sets $\cdots, A_{-2}, A_{-1}, A_0, A_1, A_2, \cdots$ such that, for each $i \in Z$, every uncountable closed subset of the x-axis intersects A_i, [7, p. 514]. Let $X = \{w\} \times \{0,1\} \cup \bigcup_{i \in Z} [P_i \cup A_i]$. If $i \in Z$, $p \in P_i$ and n is a positive integer, let $R_n(p) = D \cap P_i$ where D is the 1/n neighborhood of p in E^3. If $i \in Z$, $x \in A_i$ and n is a positive integer, let $R_n(x)$ be the set comprising (1) the bounded component of $P_{i-1} \backslash J$, where J is the circle containing x of radius 1/n and lying in $P_{i-1} \cup \{x\}$; (2) the bounded component of $P_i \backslash C$, where C is the circle containing x of radius 1/n and lying in $P_i \cup \{x\}$; and (3) $\{x\}$. If n is a positive integer and D is the 1/n neighborhood of w in E^3, then $R_n((w,0)) = \{(w,0)\} \cup [D \cap \bigcup_{i \leq -n} (P_i \cup A_{i-1})]$ and $R_n((w,1)) = \{(w,1)\} \cup [D \cap \bigcup_{i \geq n} (P_i \cup A_{i+1})]$. If, for each n, $G_n = \{R_n(p) | p \in X, i \geq n\}$, then $G_1, G_2, G_3, \cdots$ is a development for the space X with basis G_1 and has the j-link property for every j. However, by an argument similar to Jones' that his space A_∞ in [6] is not completely regular at p, we can show that X is not completely regular at (w,0) nor at (w,1) and, indeed, there is no continuous real-valued function on S such that $f((w,0)) \neq f((w,1))$. Thus, X is not submetrizable. It would be possible, using a construction similar to that of Younglove [9], to construct a space Y, using X as the first stage of the construction, having the j-link property for every j, which is a separable, locally connected and complete Moore space on which every continuous real-valued function is constant.

The space $X \setminus \{(w,0)\}$ is a submetrizable Moore space which is not completely regular at (w,1).

A space S is said to be regularly submetrizable provided that there exists a continuous 1-1 mapping of S onto a metric space M such that, for each $p \in S$ and open set 0 of S containing p, there is an open set D of S containing p such that, the closure in M of f(D) is a subset of f(0), [1].

Question 3. Is every regularly submetrizable Moore space completely regular?

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The University of Houston

Houston, Texas 77004