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## SUBCONTINUA OF FINITE UNIONS OF DENDRITES

by

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**SUBCONTINUA OF FINITE UNIONS OF DENDRITES****W. S. Mahavier****1. Introduction**

By a compactum we mean a compact subset of a metric space and by a continuum we mean a connected compactum. A dendrite is a locally connected continuum which contains no simple closed curve and a dendroid is a continuum which is arcwise connected and hereditarily unicoherent. If  $\alpha$  is an ordinal and  $p$  is a point of a compactum  $X$ , then  $X$  is said to have rim type  $\leq \alpha$  at  $p$  provided that for each  $\epsilon > 0$ , there is an open subset  $0$  of  $X$  such that  $p \in 0$ ,  $\text{diam}(0) < \epsilon$ , and the  $\alpha$ th derivative of the boundary of  $0$  in  $X$  is empty.  $X$  is said to have rim type  $\alpha$  if  $\alpha$  is the least ordinal such that  $X$  has rim type  $\leq \alpha$  at each of its points. The results of this paper were motivated by A. Lelek who asked the author if every continuum (or every dendroid) of rim type 2 could be embedded in a finite union of dendrites. It follows from a theorem of Reschovsky [3] that the union of  $n$  dendrites has rim type  $\leq n$ . We show that the Sierpinski triangular curve [1], which has rim type 1, is not embeddable in any finite union of dendrites and we construct a dendroid of rim type 2 which is not embeddable in a finite union of dendrites. A dendroid of rim type 1 is a regular curve and a dendrite [2]. Examples are given, for each  $n$ , of a continuum of rim type  $n$  which is embeddable in the union of  $2^n$  dendrites. Steenrod in [4] shows that a compact subset of  $E^n$  is a subset of the union of two arcs if and only if it contains no continuum of condensation. We introduce an extension of the concept of continuum of convergence in an attempt to characterize those dendroids which are embeddable in finite unions of dendrites.

## 2. Continua of Convergence and Embeddings in Dendrites

By a continuum of convergence of a compactum  $M$  is meant a nondegenerate continuum  $K$  lying in  $M$  which is the sequential limiting set of a sequence of mutually disjoint continua, each lying in  $M$  and none intersecting  $K$ . If  $G$  is a collection of continua in  $M$ , then  $L(G)$  denotes the collection of all nondegenerate continua  $K$  in  $M$  which are sequential limiting sets of sequences of mutually disjoint members of  $G$ , none intersecting  $K$ . Thus if  $S_0(M)$  denotes the collection of all continua in  $M$ , then  $L[S_0(M)]$  denotes the collection of all continua of convergence of  $M$ . We define inductively two sequences of collections of continua:  $C_1(M)$ ,  $C_2(M)$ , ..., and  $S_1(M)$ ,  $S_2(M)$ , ... . Let  $C_1(M) = L[S_0(M)]$ , and let  $S_1(M)$  denote the collection to which  $K$  belongs if and only if  $K$  is a nondegenerate continuum in  $M$  which is the closure of the union of the members of a subcollection of  $C_1(M)$ . For each positive integer  $n$ , let  $C_{n+1}(M) = L[S_n(M)]$ , and let  $S_{n+1}(M)$  denote the collection to which  $K$  belongs if and only if  $K$  is a nondegenerate continuum in  $M$  which is the closure of the union of the members of a subcollection of  $C_{n+1}(M)$ . By an  $n^{\text{th}}$  order convergence continuum of  $M$  we mean a member of  $C_n(M)$ . We conjecture that if  $M$  is a dendroid and for some  $n$ ,  $C_n(M) = \emptyset$ , then  $M$  is embeddable in a finite union of dendrites, and we establish the following theorem.

*Theorem.* If  $M$  is a compactum in a metric space  $S$ ,  $n$  is a positive integer, and  $M$  has an  $n^{\text{th}}$  order convergence continuum, then  $M$  is not a subset of the union of  $n$  dendrites in  $S$ .

We actually prove the following theorem for which the theorem stated above is a special case.

*Theorem.* If  $M$  is a compact subset of the metric space  $S$ ,  $n$  is a positive integer,  $L \in C_n(M)$  and  $O$  is an open set

*intersecting L, then  $0 \cap M$  is not a subset of the union of n dendrites in S.*

We first consider the case  $n = 1$ . Let  $p$  and  $q$  denote two points of  $L$ , with  $p \in 0$ . Let  $D$  denote an open set such that  $p \in D$ ,  $q \notin \bar{D}$ , and  $\bar{D} \subseteq 0$ . Let  $M_1, M_2, \dots$  denote a sequence of mutually disjoint continua in  $M$ , none intersecting  $L$ , with  $L = \lim M_i$ . With the aid of Theorems 52 and 59 of Chapter I of [2], it can be seen that there is an increasing sequence  $n_1, n_2, \dots$  of positive integers, a sequence  $p_1, p_2, \dots$  of points and a sequence  $C_1, C_2, \dots$  of continua such that (1) the sequence  $p_1, p_2, \dots$  converges to  $p$ , (2) for each positive integer  $i$ ,  $p_i \in M_{n_i} \cap D$ , (3) for each  $i$ ,  $C_i$  is the closure of the component of  $M_{n_i} \cap D$  which contains  $p_i$ , and (4)  $C_1, C_2, \dots$  has a sequential limiting set  $L'$  which is a subset of  $L$  and contains a point  $b$  of the boundary of  $D$ . Assume now that  $0 \cap M$  is a subset of a dendrite  $K$ . Then  $p$  and  $b$  are in  $K$  and since each two points of a dendrite are separated by some point of that dendrite (see Theorem 74, p. 129 of [2]), there is a point  $x$  in  $K$  such that  $K - \{x\}$  is the union of the two separated sets  $U$  and  $V$  with  $p \in U$  and  $b \in V$ .  $L' \subseteq K$  and contains both  $p$  and  $b$  and is connected so  $x \in L'$ . For each positive integer  $i$ ,  $C_i$  is a connected subset of  $K - \{x\}$ , so there is a subsequence  $C_{m_1}, C_{m_2}, \dots$  of  $C_1, C_2, \dots$  such that for each  $i$ ,  $C_{m_i} \subseteq U$  or for each  $i$ ,  $C_{m_i} \subseteq V$ . But this implies that  $L' \subseteq \bar{U}$  or  $L' \subseteq \bar{V}$ , whence  $b \in \bar{U} \cap V$  or  $p \in U \cap \bar{V}$ . This contradiction completes the argument for the case  $n=1$ . We establish the theorem by induction on the integer  $n$ . Assume that the theorem holds for each integer  $j$ ,  $1 \leq j \leq n$ , that  $M$  is a compactum, that  $L \in C_{n+1}(M)$ , that  $0$  is an open set intersecting  $L$ , and that  $0 \cap M$  is a subset of the union of the  $n+1$  dendrites  $K_1, K_2, \dots, K_{n+1}$ . Let  $M_1, M_2, \dots$  denote a sequence of mutually exclusive continua in  $S_n(M)$  with sequential limiting

set  $L$  and such that none of them intersect  $L$ . We shall show that  $(\bigcup_{i=1}^{\infty} M_i) \cap 0 \subseteq K_1$ . Assume there is, for some  $i$ , a point  $p \in (M_i \cap 0) - K_1$ . Then there is an open set  $D$  containing  $p$  but no point of  $K_1$ , and thus  $D \cap M$  is a subset of the union of the  $n$  dendrites  $K_2, K_3, \dots, K_{n+1}$ . Since  $M_i \in S_n(M)$ , there is a subcollection  $H$  of  $C_n(M)$  such that  $M_i = \overline{H^*}$ . Since  $D$  is an open set containing the point  $p$  in  $M_i$ ,  $D$  must intersect some member  $h$  of  $H$ . But  $h \in C_n(M)$  and  $D$  is an open set intersecting  $h$ , so by our inductive hypothesis  $D \cap M$  is not a subset of the union of  $n$  dendrites. It follows that  $(\bigcup_{i=1}^{\infty} M_i) \cap 0 \subseteq K_1$ . Now let  $J = L \cup \bigcup_{i=1}^{\infty} M_i$ . Then  $J$  is a compactum,  $L \in C_1(J)$  and  $0$  is an open set intersecting  $L$ , so by our inductive hypothesis,  $J \cap 0$  is not a subset of a dendrite. But  $J \cap 0 \subseteq K_1$ . This completes our argument.

It would be interesting to know if it is true that if  $M$  is a dendroid and  $n$  is a positive integer so that  $M$  contains no  $n$ th order convergence continuum, then  $M$  has rim type  $\leq n$ . If so, then this, together with the theorem above would provide another proof of Reschovsky's result for those subcontinua of finite unions of dendrites which are dendroids. On the other hand, our example in section 3 below shows that the converse is false since our example is a dendroid of rim type 2 and, for each  $n$ , it contains an  $n$ th order convergence continuum.

### 3. An Example With Rim Type 2

In this section we construct a dendroid of rim type 2 which is not embeddable in any finite union of dendrites. We note that a dendroid of rim type 1 is a regular curve and thus a dendrite (see [1], p. 283). On the other hand, the Sierpinski triangular curve ([1], p. 276) has rim type 1 and is not embeddable in a finite union of dendrites. This can be seen by an argument similar to that given below for our example.

In our description, if  $x$  and  $y$  are two points in  $E^2$ , then  $[x,y]$  denotes the closed interval from  $x$  to  $y$  and  $|x - y|$  denotes the Euclidean distance from  $x$  to  $y$ . By a wedge we mean a sector of a semicircle in  $E^2$ . More accurately,  $W$  is a wedge if and only if there are three non-collinear points  $a, b$  and  $c$  in  $E^2$  with  $|b - a| = |c - a|$  such that  $W$  is the convex 2-cell in  $E^2$  whose boundary is the union of  $[a,b]$ ,  $[a,c]$ , and the short arc from  $b$  to  $c$  on the circle with center  $a$  and radius  $|b - a|$ . The points  $a, b$  and  $c$  are called vertices of  $W$ ,  $a$  is called its apex, and the intervals  $[a,b]$  and  $[a,c]$  are called its sides. One of the sides of  $W$ , called its base, can be taken onto the other side by a counterclockwise rotation of the plane of less than  $\pi$  radians. For each wedge  $W$  in  $E^2$ , we describe a collection  $Q(W)$  of wedges lying in  $W$ . Let  $[a,b]$  denote the base of  $W$  and let  $[a,b_0]$  denote its other side. Let  $b_1, b_2, \dots$  denote a sequence of points on the circular arc in the boundary of  $W$  which converge to  $b$  and where  $|b - b_{n-1}| = 2|b - b_n|$  for each positive integer  $n$ . For each positive integer  $n$ , partition the interval  $[a, b_n]$  into  $2^n$  non-overlapping subintervals each of length  $|b - a|/2^n$ . Let  $p(n,0), p(n,1), \dots, p(n,2^n)$  denote the endpoints of the intervals in this partition, the notation chosen so that for  $1 \leq m \leq 2^n$ ,  $|p(n,m) - a| = m|b - a|/2^n$ . Thus  $p(n,0) = a$  and  $p(n,2^n) = b_n$ . For each positive integer  $n$  and each positive integer  $m \leq 2^n$ , let  $W(n,m)$  denote the wedge having  $p(n,m - 1)$  as its apex, the interval  $[p(n,m - 1), p(n,m)]$  as its base and whose apex angle is half that of the wedge with vertices  $a, b_n$  and  $b_{n-1}$ . Finally we let  $Q(W)$  denote the collection of all wedges  $W(n,m)$  for all positive integers  $n$  and  $m$  with  $m \leq 2^n$ .

To describe our example we let  $M_0$  denote a wedge of base length  $1/2$ , and define inductively a sequence  $Q_1, Q_2, \dots$  of collections of wedges such that  $Q_1 = Q(M_0)$  and, for each positive integer  $n$ ,  $Q_{n+1} = \bigcup_{W \in Q_n} Q(W)$ . Then for each  $n$ , let  $M_n = \overline{Q_n^*}$

and let  $M = \bigcup_{n=1}^{\infty} M_n$ .

For each  $n$ ,  $M_n$  is a compact continuum which does not separate  $E^2$ , so  $M$  is a compact, non-separating plane continuum. Each member of  $Q_n$  is of diameter not more than  $1/2^n$  and each point of  $M_n$  is either in the interior of a wedge in  $Q_n$  or is a boundary point of  $M_n$  in  $E^2$ . It follows that  $M_n$  contains no 2-cell of diameter more than  $1/2^n$ , and thus  $M$  contains no 2-cell. To see that no subcontinuum of  $M$  separates  $E^2$ , let  $K$  denote a subcontinuum of  $M$  and assume that  $E^2 - K$  is the union of two separated sets  $U$  and  $V$ . If  $n$  is a positive integer,  $M_n$  does not separate  $E^2$  so one of  $U$  or  $V$  is a subset of  $M_n$ . This implies that one of  $U$  or  $V$  contains no 2-cell but each is open in  $E^2$ . Thus  $K$  is unicoherent and  $M$  is hereditarily unicoherent. To establish that  $M$  is arcwise connected, we shall indicate a construction of an arc from the apex  $p_0$  of  $M_0$  to an arbitrary point  $x \in M - \{p_0\}$ . Assume that there is a positive integer  $n$  such that  $x \in M_n - Q_n^*$ . Since for  $n > 1$ ,  $M_n - Q_n^*$  is the union of the bases of all wedges in  $\{M_0\} \cup \bigcup_{i=1}^{n-1} Q_i$ , it is easily seen that there is an arc in  $M_n - Q_n^*$  which is the union of at most  $n - 1$  straight line intervals. Next assume that for each positive integer  $n$ ,  $x$  is in the interior of some wedge in  $Q_n$ . If  $p_n$  denotes the apex of the wedge in  $Q_n$  which contains  $x$ , then the sequence  $p_1, p_2, \dots$  converges to  $x$  and  $(\bigcup_{i=0}^{\infty} [p_i, p_{i+1}]) \cup \{x\}$  is an arc from  $p_0$  to  $x$ . We next show that  $M$  has rim type 2. Let  $x \in M$  and first assume that for each positive integer  $n$ ,  $x$  is in the interior of some wedge  $W$  in  $Q_n$ . If  $a$  and  $b$  denote the endpoints of the base of  $W$ , then  $(W - \{a, b\}) \cap M$  is open in  $M$ , contains  $x$ , is of diameter not more than  $1/2^n$ , and has a boundary in  $M$  which consists of only the two points  $a$  and  $b$ . So  $M$  has rim type 1 at  $x$ . If  $x$  is not, for each  $n$ , in the interior of a wedge in  $Q_n$ , then there exists an integer  $n$  such that  $x$  is on the base of a wedge  $W$  in  $Q_n$ . First assume that  $x$  is not an

endpoint of the base of  $W$ . In this case  $x$  is in the interior with respect to  $M$  of  $W$ , and  $W \cap M$  is homeomorphic to  $M$  so it suffices to consider the case where  $x$  is on the base of  $M_0$  but is not an endpoint of the base of  $M_0$ . Let each of  $n$  and  $k$  denote a positive integer with  $k \leq 2^n$ , let  $C(n,k)$  denote the circle with center  $p_0$  and radius  $k/2^n$  and let  $W_n$  denote the wedge with the same base as  $M_0$  and whose apex angle is  $\theta/2^n$  where  $\theta$  is the apex angle of  $M_0$ . It can be seen from the construction of  $M$  that  $C(n,k) \cap W_n \cap M$  is a convergent sequence of points whose limit is on the base of  $M_0$ . For each positive integer  $n$ , there is a nonnegative integer  $m$  such that  $(m/2^n) < |x - p_0| < (m+2)/2^n$ . Let  $O_n$  denote the set of all points of  $W_n \cap M$  which are interior to  $C(n,m)$  and exterior to  $C(n,m+2)$ . Then  $O_n$  is an open subset of  $M$  containing  $x$  whose boundary in  $M$  consists of at most two convergent sequences of points, and, since  $\text{diam}(O_n) \rightarrow 0$  as  $n \rightarrow \infty$ , we have that  $M$  has rim type 2 at  $x$ . Note that the same argument applies in case  $x$  is the endpoint of the base of  $M_0$  different from  $p_0$ . We next consider the special case  $x = p_0$ . For each positive integer  $n$ , let  $C_n$  denote the circle with center  $p_0$  and radius  $1/2^n$ , and let  $V_n$  denote the wedge in  $Q_1$  having apex  $p_0$  and of diameter  $1/2^n$ . We shall show inductively that  $C_n \cap M$  has only finitely many limit points, the number of limit points increasing with  $n$ . First note that  $C_1 \cap M = C_1 \cap M_1$  and has only one limit point. For each positive integer  $n$ , there is a homeomorphism  $h_n: V_n \cap M$  onto  $M$  such that for each  $x$  in  $V_n \cap M$ ,  $|h_n(x) - p_0| = 2^n |x - p_0|$ . Consider now  $C_2 \cap M$ . We have that  $C_2 \cap M \cap V_1$  is homeomorphic to  $C_1 \cap M$  and thus has only one limit point. And, from the construction it follows that  $C_2 \cap (M_1 - V_1) = C_2 \cap M_1 - V_1$  and is convergent sequence of points. Continuing we see that if  $n$  is a positive integer and  $j$  is a positive integer,  $j < n$ , then  $C_n \cap V_j \cap M$  is homeomorphic to  $C_{n-j} \cap M$  and thus



$C_n \cap M \cap (\bigcup_{i=1}^{n-1} V_i)$  has only finitely many limit points. Further

$C_n \cap (M - \bigcup_{i=1}^{n-1} V_i) = C_n \cap (M_{n-1} - \bigcup_{i=1}^{n-1} V_i)$  and is a convergent sequence of points. The only remaining case to consider is that

in which  $x$  is a common vertex of two wedges in  $Q_n$  for some  $n$ .

It can be shown that  $M$  has rim type 2 at such points by combining the methods used earlier for the case  $x = p_0$  and the case where  $x$  is the other endpoint of the base  $M_0$ .

We next show that if  $M$  denotes the example described above or the Sierpinski triangular curve then  $M$  is not embeddable in a finite union of dendrites. Assume the contrary and let  $n$  denote the least positive integer such that  $M$  is embeddable in  $n$  dendrites. Let  $K_1, K_2, \dots, K_n$  denote  $n$  dendrites and let  $h$  denote a homeomorphism from  $M$  into  $S = \bigcup_{i=1}^n K_i$ . Clearly  $n \neq 1$ , so there is a point  $p$  of  $M$  not in  $K_1$  and an open set  $O$  in  $S$  containing  $p$  but no point of  $K_1$ . Since each open subset of  $M$  contains a homeomorphic image of  $M$ , then  $O \cap h(M)$  must contain a homeomorphic image of  $M$  which is embedded in the union of the  $n - 1$  dendrites  $K_2, K_3, \dots, K_{n-1}$ .

#### 4. Examples of Higher Rim Types

In this section we briefly describe, for each positive integer  $n$ , a continuum  $K_n$  having rim type  $n + 1$  and which is the union of  $2^n$  dendrites. We let  $K_1$  denote the continuum described in [1], p. 268, and let  $F_1$  and  $F_2$  denote the dendrites also described in [1] whose union is  $K_1$ . We assume that  $F_1$  contains the interval in  $E^2$  from the point  $(0,0)$  to  $(0,1)$ . For each rectangular disc  $R = [a,b] \times [c,d]$  in  $E^2$  we let  $K_1(R)$  denote the image of  $K_1$  under the natural linear homeomorphism  $h$  of  $[0,1] \times [0,1]$  onto  $R$  given by  $h(x,y) = (a(1-x) + bx, c(1-x) + dx)$ . Similarly we define  $F_i(R) = h(F_i)$  for  $1 \leq i \leq 2$ . For each integer  $n > 0$ , and each integer  $m$ ,  $0 \leq m < 2^n$ , let  $R(n,m) = [m/2^n, (m+1)/2^n] \times [1/2^{n-1}, 3/2^n]$ . Next define

$K_2 = K_1 \cup \bigcup_{\substack{n>0 \\ 0 \leq m < 2^n}} K_1[R(n,m)]$ .  $K_2$  has rim type 3 and is the union

of the 4 dendrites given by:

$$F_{i_1, i_2} = F_{i_1} \cup \bigcup_{\substack{n>0 \\ 0 \leq m < 2^n}} F_{i_2}[R(n,m)].$$

We proceed inductively to define a sequence  $K_1, K_2, K_3, \dots$  of continua such that for each  $n$ ,  $K_n$  has rim type  $n + 1$ . For each positive integer  $j$ , let  $K_j[R(n,m)]$  denote the image of  $K_j$  under the natural linear homeomorphism of  $[0,1] \times [0,3/2]$  onto  $R(n,m)$ , and let  $K_{j+1} = K_1 \cup \bigcup_{\substack{n>0 \\ 0 \leq m < 2^n}} K_j[R(n,m)]$ . We also define inductively,

for each positive integer  $n$ , a collection of  $2^n$  dendrites whose union is  $K_n$ . For each  $n$ -term sequence  $i_1, i_2, \dots, i_n$  each term of which is 1 or 2, we set

$$F_{i_1, i_2, \dots, i_n} = F_{i_1} \cup \bigcup_{\substack{n>0 \\ 0 \leq m < 2^n}} F_{i_2, \dots, i_n}[R(n,m)]$$

where  $F_{i_2, \dots, i_n}[R(n,m)]$  is the image of  $F_{i_2, \dots, i_n}$  under the natural linear homeomorphism of  $[0,1] \times [0,3/2]$  onto  $R(n,m)$ .

It is easily established inductively that  $K_n$  is the union of the  $2^n$  dendrites  $\{F_a\}$  for all  $n$ -term sequences  $a$ , each term of which is 1 or 2.

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