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## Research Announcement:

# SETS OF DISTANCES AND MAPPINGS OF CERTAIN CONTINUA

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*Research Announcement*

# SETS OF DISTANCES AND MAPPINGS OF CERTAIN CONTINUA

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Given a function f defined on a metric space X, we denote by $\Delta_f$ the set of distances

$$\Delta_f = \{\text{dist}(x,x'): x,x' \in X, f(x) = f(x')\}.$$

By a *continuum* we mean a connected compact metric space. If $f: X \to Y$ is a continuous mapping of a compact metric space X which lowers the dimension of X, then there exists a point $y_0 \in Y$ such that the set $f^{-1}(y_0)$ is positive-dimensional. Thus, in this case, $f^{-1}(y_0)$ contains a non-degenerate continuum, and $\Delta_f$ contains an interval $[0,a]$, where $a > 0$. The latter property can, however, be possessed also by those mappings which do not necessarily lower the dimension of the space. Then, in most cases, the distances of $\Delta_f$ filling up an interval may have to be selected between pairs of points belonging to many sets $f^{-1}(y)$ instead of just one. For example, it is known [6] that if $f: S^n \to R^n$ is a continuous mapping of the standard n-sphere $S^n$ lying in the Euclidean (n+1)-space into the Euclidean n-space $R^n$, where n is an even positive integer, then $\Delta_f$ contains an interval $[0,a]$, where $a = a(S^n) > 0$. The same conclusion holds [4] for real-valued continuous functions $f: T \to R$ defined on any simple triod T. More precisely, the right end-point of an interval contained in $\Delta_f$ can be taken to be $a = \sigma(T)$ the so-called "span" of T as introduced in [1], and the simple triod T can be replaced by any unicoherent locally connected continuum.

The aim of this paper is to announce several results concerning the magnitude of the set $\Delta_f$ for continuous mappings $f: X \to Y$ of continua that satisfy some conditions. The concept of the span will be used as well as some other related concepts. The detailed proofs and more discussion will be published in

[2] and [3].

Let X be a connected metric non-empty space. The standard projections of the product $X \times X$ onto X will be denoted by $p_1$ and $p_2$, that is $p_1(x,x') = x$ and $p_2(x,x') = x'$ for $(x,x') \in X \times X$. We define the *surjective span* $\sigma^*(X)$ [the *surjective semispan* $\sigma_0^*(X)$] of X to be the least upper bound of the set of all real numbers $\alpha \geq 0$ with the following property: there exists a connected set $C_\alpha \subset X \times X$ such that $\alpha \leq \text{dist}(x,x')$ for $(x,x') \in C_\alpha$ and $p_1(C_\alpha) = p_2(C_\alpha) = X$ [$p_1(C_\alpha) = X$, respectively]. Clearly, we have the inequalities

$$0 \leq \sigma^*(X) \leq \sigma_0^*(X) \leq \text{diam } X,$$

where the diameter of X can be $+\infty$ in case X is unbounded. The *span* $\sigma(X)$ and the *semispan* $\sigma_0(X)$ of X can now be defined by the formulae

$$\sigma(X) = \text{Sup}\{\sigma^*(A): A \subset X, A \neq \emptyset \text{ connected}\},$$

$$\sigma_0(X) = \text{Sup}\{\sigma_0^*(A): A \subset X, A \neq \emptyset \text{ connected}\},$$

whence

$$0 \leq \sigma(X) \leq \sigma_0(X) \leq \text{diam } X,$$

and also $\sigma^*(X) \leq \sigma(X)$ and $\sigma_0^*(X) \leq \sigma_0(X)$. There exist examples of fine continua, such as simple triods [3] and simple "4-ods" [5], showing that both the surjective semispan $\sigma_0^*(X)$ and the span $\sigma(X)$ can be strictly less than the semispan $\sigma_0(X)$. The semispan then turns out to be the greatest of the four quantities $\sigma(X)$, $\sigma^*(X)$, $\sigma_0(X)$, and $\sigma_0^*(X)$ discussed here. Of course, it is possible that all these numbers are equal to zero, e.g., for X being an arc.

Let X be a continuum. We say that X is *arc-like* (or *P-unicoherent*) provided, for each $\varepsilon > 0$, there exists a finite open cover of X whose elements have diameters less than $\varepsilon$ and whose nerve, if non-degenerate, is an arc (or a unicoherent polyhedron, respectively).

*Theorem 1. If* $f: X \to Y$ *is a continuous mapping of a P-unicoherent continuum* $X$ *onto an arc-like continuum* $Y$, *then*

$$[0,\sigma(X)] \subset \Delta_f.$$

(See [2].)

*Theorem 2. Let* $\tau = \sigma^*, \sigma_0^*$. *If* $f: X \to Y$ *is a continuous mapping of a P-unicoherent continuum* $X$ *onto a continuum* $Y$ *with* $\tau(Y) = 0$, *then*

$$[0,\tau(X)] \subset \Delta_f.$$

*Theorem 3. Let* $\tau = \sigma, \sigma^*, \sigma_0, \sigma_0^*$. *If* $f: X \to Y$ *is a continuous mapping of a tree-like continuum* $X$ *onto a continuum* $Y$ *with* $\tau(Y) = 0$, *then*

$$[0,\tau(X)] \subset \Delta_f.$$

(See [3].)

*Corollary. If* $f: X \to Y$ *is a continuous mapping of a tree-like continuum* $X$ *onto an arc-like continuum* $Y$, *then*

$$[0,\sigma_0(X)] \subset \Delta_f.$$

This corollary is stronger than the result of [4] applied to tree-like continua. Indeed, as we have mentioned, there exists a simple 4-od X (Example 2 of [5]) such that $\sigma(X) < \sigma_0(X)$ (cf. [2] and [3]).

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