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Research Announcement:

ON FREE TOPOLOGICAL GROUPS AND FREE PRODUCTS OF TOPOLOGICAL GROUPS

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ON FREE TOPOLOGICAL GROUPS AND FREE PRODUCTS OF TOPOLOGICAL GROUPS

(summary of results to appear elsewhere)

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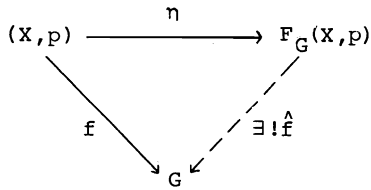
In attempting to characterize the epimorphisms in the category of Hausdorff topological groups, one is led to investigating certain quotients of the free product $G \amalg G$ of a Hausdorff topological group with itself. In particular, one wants to know if the amalgamated free product $G \amalg_B G$, for a closed subgroup B, is Hausdorff, or if its Hausdorff reflection is "sufficiently large." To do this it seems useful to obtain information about the topological structure of $G \amalg G$, or more generally of $G \amalg H$ for Hausdorff topological groups G and H.

Given any two topological groups (not necessarily Hausdorff) G and H, their coproduct $G \amalg H$ in the category of topological groups is $G * H$ (their free product in the category of groups) with the finest topology compatible with the group structure making the injections $G \to G \amalg H \leftarrow H$ continuous (Wyler). The subgroup K of $G \amalg H$ generated by the reciprocal commutators $[g,h] = ghg^{-1}h^{-1}$ is a normal subgroup and is freely generated. It is not difficult to show that every element of $G \amalg H$ has a unique representation ghc with $c \in K$, and if G and H are Hausdorff then K is closed in $G \amalg H$. Theorem 4 below shows that if G and H are Hausdorff and if K can be given a suitable topology, then $G \amalg H$ is Hausdorff.

Since K is freely generated we begin with a short investigation of Graev free groups.

Defn: The *Graev free topological group* over a pointed space (X,p) consists of a topological group $F_G(X,p)$ and a base point preserving continuous function $\eta:(X,p) \to F_G(X,p)$ with the usual unique factorization property



(the base point of a group is the identity, f is continuous and preserves base points, the induced $\hat{f}$ is a continuous group homomorphism, and the diagram commutes).

Theorem (Wyler)*: The underlying group of* $F_G(X,p)$ *is the free group on* $X\backslash\{p\}$*, and* $F_G(X,p)$ *has the finest topology compatible with the group structure such that* $\eta: X \to F_G(X,p)$ *is continuous.*

Theorem (Ordman)*: If* X *is a* k_ω*-space*[1] *then* $F_G(X,p)$ *has the weak topology generated by the subsets* $[F_G(X,p)]_n$ = *words of reduced length* $\leq$ n.

Theorem 1*:* $F_G(X,p)$ *is Hausdorff if and only if* X *is functionally Hausdorff.*

Theorem 2*:* $F_G(X,p)$ *contains* X *as a closed subspace if and only if* X *is Tychonoff.*

Theorem 3*: If* X *is Tychonoff and* Y *is a closed subspace of* X *containing the base point then the subgroup of* $F_G(X,p)$ *generated by* Y *is closed.*

We now turn to considering the topological structure of $G \amalg H$ for Hausdorff groups G and H.

Theorem 4*: In order that* $G \amalg H$ *have a Hausdorff group*

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[1]X is a k_ω-space if it is the weak sum of countably many compact Hausdorff spaces.

topology it is necessary and sufficient that $K \lhd G \amalg H$ *have a Hausdorff group topology, no finer than, but comparable to, the topology of* $F_G(G \wedge H, \bar{e})$[2] *such that* $\psi: (G \times H) \times K \to K$, *where* $\psi(g,h,c) = ghch^{-1}g^{-1}$, *is continuous.*

Note: the need for the continuity of ψ shows up a number of times in the proof of this theorem, of which the simplest example is

$$(ghc)^{-1} = g^{-1}h^{-1}(hgh^{-1}g^{-1})(ghc^{-1}h^{-1}g^{-1})$$
$$= g^{-1}h^{-1}[g,h]^{-1}\psi(g,h,c^{-1})$$

Ordman's Theorem above says roughly "If X is a k_ω-space and something is true for words of length $\leq n$, for all n, then it is true for $F_G(X,p)$. Thus

Corollary (Katz)*: If* G *and* H *are* k_ω*-spaces, then* $G \amalg H$ *is Hausdorff.*

The essential observation in the proof is that for all n $\psi_n: (G \times H) \times [F_G(G \wedge H, \bar{e})]_n \to F_G(G \wedge H, \bar{e})$ is continuous.

We conclude with a few observations. The first answers a question of Morris, Ordman, and Thompson in the negative.

Observation 1: $G \wedge H$ and $[G,H] \subseteq G \amalg H$ need not have the same topology: In $X = (\omega_1+1) \times \omega_1$ the two closed sets $A = \{(x,x) | \omega_0 \leq x < \omega_1\}$ and $B = ((\omega_1+1) \times \{1\}) \cup (\{\omega_1\} \times \omega_1)$ cannot be separated by open sets. Let $G = F_G(\omega_1+1, \omega_1)$ and $H = F_G(\omega_1, 1)$. Then X is a closed subset of $G \times H$, A is closed in $G \times H$ and $B = X \cap ((G \times \{e_H\}) \cup (\{e_G\} \times H))$. Hence A and $(G \times \{e_H\}) \cup (\{e_G\} \times H)$ cannot be separated by open sets. It follows that $G \wedge H$ is not even regular, and thus cannot be a subspace of $G \amalg H$.

Observation 2: The method of proof used in the corollary

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[2] $G \wedge H$ is the quotient of $G \times H$ obtained by collapsing $(G \times \{e_H\}) \cup (\{e_G\} \times H)$ to a point, this collapsed point is denoted by $\bar{e}$.

above cannot be used in the general case since $F_G(\mathbf{Q},0)$ does not have the weak topology generated by the words of length $\leq n$, and $\mathbf{Q} = \mathbf{Q} \wedge \{0,1\}$.

Obersvation 3: We cannot even replace $F_G(G \wedge H,\bar{e})$ by $F_G(\beta(G \wedge H),\bar{e})$ to obtain a proof of the general case since $(\mathbf{Q} \times \mathbf{Q}) \times F_G(\beta(\mathbf{Q} \wedge \mathbf{Q}),\bar{0})$ does not have the weak topology generated by the subsets $(\mathbf{Q} \times \mathbf{Q}) \times [F_G(\beta(\mathbf{Q} \wedge \mathbf{Q}),\bar{0}]_n$.

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