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A SURVEY OF MAXIMAL TOPOLOGICAL SPACES

by

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1. Introduction

For a topological property R and a set X, the collection R(X) of all topologies on X with property R is partially ordered by set containment. A member τ of R(X) is maximal R (R-maximal) if τ is a maximal element of R(X). Characterizing R-maximal spaces and determining which members of R(X) are strongly R (that is, for which topologies there exist finer maximal R topologies) are the main areas of interest in the study of maximal topologies. Properties of minimal R topologies are also studied and a compilation of known results has been done for minimal spaces [5]. It is our intention in this article to include all known results in the study of maximal spaces including related results in the theory of topological expansions. We realize that we may have missed some references but have included all those known to us in the bibliography.

The first result concerning minimal or maximal topologies is that every one-to-one continuous mapping of a Hausdorff compact space into a Hausdorff space is a homeomorphism (i.e., a Hausdorff compact space is minimal Hausdorff). This result has been credited to A. S. Parhomenko [41] based upon a German summary to a Russian article. A recent translation [42] of the article reveals that Parhomenko credits either P. Alexandrov and H. Hopf [1] or F. Hausdorff [26] with the result.

In 1943, E. Hewitt [27] showed that a compact Hausdorff space is maximal compact as well as minimal Hausdorff. While Hewitt did not state his result in these terms, he did use the term "maximal" in the sense it is used today. In 1947, R. Vaidyanathaswamy [57] observed the same result as Hewitt and questioned the existence of maximal compact spaces which are not Hausdorff. (He used the term "minimal compact.") In 1948, A. Ramanathan [44] characterized the maximal compact spaces as those compact spaces in which the compact subsets are precisely the closed sets, and exhibited a maximal compact space which is not Hausdorff, thus answering Vaidyanathaswamy's question. H. Tong [54] and V. K. Balachandran [4] also obtained examples of maximal compact spaces which are not Hausdorff.

In 1963, N. Smythe and C. A. Wilkens [48] characterized maximal compact spaces as A. Ramanathan had, and gave an example of a maximal compact space which is not Hausdorff but for which a finer minimal Hausdorff space exists. In 1964, N. Levine [36] introduced the concept of simple expansion and in 1965 [37] discussed spaces in which the compact and closed sets are the same (maximal compact spaces) proving that the product of a maximal compact space with itself is maximal compact if and only if the space is Hausdorff.

S. Ikenaga and I. Yoshioka [28] discussed preservation of topological properties under expansions in 1965 and this work was supplemented by C. J. Borges, Jr. [7] in 1967. J. P. Thomas studied maximal spaces not possessing certain separation properties in 1967 [51] and in 1968 published his paper on maximal connectedness in which he asked whether or not maximal connected Hausdorff spaces existed [52]. This

paper spurred research in maximal topologies and since this time many papers and theses have appeared investigating maximal topologies and most of these appear in the bibliography. The question has recently been answered affirmatively [25].

2. Techniques and Methods

The method used in constructing finer topological spaces from a given topological space is the simple expansion. This concept was first introduced as a construction tool by N. Levine [36] in 1964 although a similar concept had been discussed by E. Hewitt in 1943 [27]. We shall use Hewitt's terminology rather than Levine's term "simple extension."

Definition 2.1. If τ_1 and τ_2 are topologies on a set X, τ_2 is an *expansion* of τ_1 if $\tau_1 \subset \tau_2$ (τ_2 is said to be *finer* (stronger) than τ_1 or τ_1 is coarser (weaker) than τ_2). An expansion τ_2 of τ_1 is a *simple expansion* of τ_1 if there is a subset A of X such that $\tau_1 \cup \{A\}$ is a subbase for τ_2.

In other words, τ_2 is a simple expansion of τ_1 if there is $A \subseteq X$ such that $\tau_2 = \{U \cup (V \cap A) \mid U, V \in \tau_1\}$. The simple expansion of τ by A shall be denoted $\tau(A)$.

Theorem 2.1 [36]. *In the following,* (X,τ) *is a topological space,* $A, B \subseteq X$, $\tau^* = \tau(A)$.

(a) $int_{\tau^*}B = int_{\tau}B \cup int_{\tau|A}(B \cap A)$

(b) $cl_{\tau^*}B = cl_{\tau}B \cap ((X-A) \cup (A \cap cl_{\tau}(B \cap A))$

(c) $(A, \tau|A) = (A, \tau^*|A)$

$(X-A, \tau|(X-A)) = (X-A, \tau^*|(X-A))$

(d) $cl_{\tau}(B \cap A) = cl_{\tau^*}(B \cap A)$

(e) A *is closed in* τ^* *if and only if* A *is closed in* τ.

(f) If F *is closed in* τ *or* τ^*, *then* F $\cap$ A *is closed in* (A, $\tau|$A) *and* F $\cap$ (X-A) *is closed in* (X-A, $\tau|$(X-A)).

Definition 2.2. If (X,τ) is a topological space, and $\mathcal{J} = \{A_\alpha \subset X | \alpha \in \Lambda\}$ is a collection of subsets of X, then the topology τ^* on X which is the coarsest topology on X which is finer than $\tau(A_\alpha)$ for each $\alpha \in \Lambda$ shall be denoted by $\tau(\mathcal{J})$. If $\mathcal{J}$ is a topology on X, then $\tau(\mathcal{J}) = \mathcal{J}(\tau) = \tau \vee \mathcal{J}$.

Theorem 2.2 [50]. *If* (X,τ) *is a topological space,* A, $B \subseteq X$, *then*

(a) $\tau(\{A,B\}) = (\tau(A))(B)$;

(b) $\tau(A) \subset \tau(B)$ *if and only if* $(bd_\tau A) \cap A \subset (bd_\tau B) \cap B$ *and there is* $U \in \tau$ *such that* $U \cap B = A \cap B$;

(c) $\tau(A) = \tau(B)$ *if and only if either of the following hold:*

(i) $(bd_\tau A) \cap A = (bd_\tau B) \cap B$ *and there are* U, V $\in \tau$ *such that* $U \cap B = A \cap B = V \cap A$.

(ii) $(bd_\tau) \cap A = (bd_\tau B) \cap B$ *and* $A \cap cl_\tau(B-A) = \phi = B \cap cl_\tau(A-B)$.

Theorem 2.3 [50]. *If* (X,τ) *is a topological space,* $\mathcal{J} = \{A_\alpha \subseteq X | \alpha \in \Lambda\}$, *then*

(a) $\tau(\mathcal{J})$ *is equivalent to a well-ordered succession of simple expansions;*

(b) $\tau(\mathcal{J}) = \tau(\cup\{A_\alpha | \alpha \in \Lambda\})$ *if and only if* $A_{\alpha_1} \cap cl_\tau(\cup\{A_\alpha | \alpha \in \Lambda - \{\alpha_1\}\}) = \phi$ *for every* $\alpha_1 \in \Lambda$.

Theorem 2.4 [50]. *If* τ_1 *and* τ_2 *are topologies on* X *then* $\tau_1 \subseteq \tau_2$ *if and only if there is a family* $\mathcal{J}$ *of subsets of* X *such that* $\tau_2 = \tau_1(\mathcal{J})$.

3. Preservation Under Expansions

In this section, we report results concerning preservation of topological properties under expansions. This is of importance since this is an underlying concept in determining maximal topologies for given topological properties.

Theorem 3.1. *(a)* [36] *If* (X,τ) *is* T_i *then* $(X, \tau(A))$ *is* T_i *for all* $A \subseteq X$, $i = 0,1,2$.

(b) [36] *If* (X,τ) *is* R $\in$ {*compact, countably compact, Lindelöf*} *then* $(X, \tau(A))$ *is* R *if and only if* $(X-A, \tau|(X-A))$ *is* R.

(c) [36] *If* (X,τ) *is second countable, then* $(X, \tau(A))$ *is second countable.*

(d) [36] *If* (X,τ) *is separable then* $(X, \tau(A))$ *is separable if and only if* $(A, \tau|A)$ *is separable.*

(e) [7] *If* (X,τ) *is* R $\in$ {*separable, second countable*} *and* $(A_n, \tau|A_n)$ *is* R, *then* $(X, \tau(\{A_n | n \in |N\}))$ *is* R.

(f) [14] *If* (X,τ) *is* R $\in$ {K-*dense, caliber*-K, *Sanin, countable chain condition*}, *and* $(A, \tau|A)$ *is* R, *then* $(X, \tau(A))$ *is* R.

Definition 3.1. A subset A of a topological space (X,τ) is R-*open* if $A \in \tau|cl_\tau A$.

Definition 3.2. A subset A of a topological space (X,τ) is *locally closed* if A is the intersection of an open set and a closed set.

Theorem 3.2. *Let* (X,τ) *be a regular space,* $A \subseteq X$. *The following are equivalent.*

(a) $(X, \tau(A))$ *is regular.*

(b) [47, 28, 7] $(cl_{\tau}A) - A$ *is closed in* τ.

(c) [47, 28] A *is* R-*open.*

(d) [47] A *is locally closed.*

Theorem 3.3. *If* (X,τ) *is a regular space,* $A, B \subseteq X$ *then*

(a) [36] $(X, \tau(A))$ *is regular if* A *is closed.*

(b) [36] *If* A *is dense in* X, $A \neq X$, *then* $(X, \tau(A))$ *is not regular.*

(c) [28] *If* $(X, \tau(A))$ *and* $(X, \tau(B))$ *are regular then* $(X, \tau(A \cap B))$ *is regular.*

(d) [50] *If* (X, τ^*) *is regular then* $(X, \tau \vee \tau^*)$ *is regular.*

Theorem 3.4 [47]. *If* R *is a topological property such that*

(i) R *implies regularity*

(ii) R *is closed hereditary, and*

(iii) R *is preserved under finite unions of closed sets,*

then $(X, \tau(A))$ *has property* R *if* $(X, \tau(A))$ *is regular and* $(X-A, \tau(X-A))$ *has property* R.

Theorem 3.5 [47]. *If* Q *is a topological property such that*

(i) Q *implies regularity*

(ii) Q *is hereditary on arbitrary subsets,*

then $(X, \tau(A))$ *has property* Q *if and only if* $(X, \tau(A))$ *is regular.*

Theorem 3.6 [28]. *If* P *is a topological property such that*

(i) P *is open (closed) hereditary*

(ii) for A, B *separated such that* $(A, \tau|A)$ *and* $(B, \tau|B)$ *has property* P,

then $(A \cup B, \tau|(A \cup B))$ *has property* P. *For* $A \subseteq X$ *such that* $X - A \in \tau$, $(X, \tau(A))$ *has* P *if and only if* $(A, \tau|A)$, $(X-A, \tau|(X-A))$ *has property* P.

While many topological properties satisfy the properties of Theorem 3.6, connectedness does not.

Theorem 3.7 [7]. *In a* T_1*-space* (X,τ), $(X, \tau(A))$ *has property* P $\in$ {*complete regularity, normality, collectionwise normality, paracompactness, stratifiability, or metrizability*} *if and only if* (X,τ) *has property* P, $(X, \tau(A))$ *is regular and* $(X-A, \tau(X-A))$ *is normal for all except complete regularity.*

Theorem 3.8 [36]. *If* (X,τ) *is completely regular (Tychonoff),* $A \notin \tau$ *but* $X - A \in \tau$ *then* $(X, \tau(A))$ *is completely regular (Tychonoff).*

Theorem 3.9 [36]. *If* (X,τ) *is normal and* $X - A \in \tau$, *then* $(X, \tau(A))$ *is normal if and only if* $(X-A, \tau|(X-A))$ *is normal.*

Theorem 3.10 [46]. *If* (X,τ) *is Hausdorff and* P $\in$ {*regularity, complete regularity, normality, paracompactness, stratifiability, metrizability, metacompactness*}, *then* $(X, \tau(A))$ *has* P *if and only if* A *is locally closed and* $(X-A, \tau|(X-A))$ *has property* P.

Theorem 3.11 [7]. *In the following all spaces are* T_1.

(a) If (X,τ) *is paracompact regular, then* $(X, \tau|A))$ *is paracompact regular if and only if* $(X, \tau(A))$ *is*

regular and $X - A$ *is paracompact in* (X,τ).

(b) If (X,τ) *has* $P \in \{$*complete regularity, hereditary normality, perfectly normal, hereditary paracompactness, stratifiability, metrizability*$\}$ *then* (X,τ) *has* P *if and only if* $(X, \tau(A))$ *is regular.*

(c) If (X,τ) *has* $P \in \{$*normality, Lindelöf, compactness, countable compactness*$\}$ *then* $(X, \tau(A))$ *has* P *if and only if* $(X, \tau(A))$ *is regular and* $(X-A, \tau|(X-A))$ *has* P.

Theorem 3.12. *If* (X,τ) *is connected and* $A \subseteq X$, *then* $(X, \tau(A))$ *is connected if*

(a) [36, 45] A *is a connected dense subset of* X;

or (b) [25, 24] A *is a connected dense subset of* $U \in \tau$;

or (c) [7] $X - A \notin \tau$ *and* $(A, \tau|A)$ *and* $(X-A, \tau|(X-A))$ *are connected*

or (d) [24] $(X-A, \tau|(X-A))$ *is connected and no component of* A *is* τ*-closed.*

or (e) [24] A *is connected and no union of components of* $X - A$ *is open.*

or (f) [24] A *has a connected component and* $X - A \in \tau$.

or (g) [24] *and only if there exists no nontrivial* $C \subseteq X$ *such that*

(i) $C \cap A \in \tau|A$

(ii) $(X-C) \cap A \in \tau|A$

(iii) $C \cap (X-A) \in \tau|(X-A \cup X-C)$

(iv) $(X-C) \cap (X-A) \in \tau|((X-A) \cup C)$

Theorem 3.13 [7]. *If* (X,τ) *is pathwise connected,* $X - A \notin \tau$ *and* $(cl_\tau A, \tau|cl_\tau A)$ *and* $(X-A, \tau|(X-A)$ *are pathwise*

connected, then $(X, \tau(A))$ *is pathwise connected.*

Theorem 3.14 [46]. *If* $n \in N$, $P \in$ {*regularity, complete regularity, normality, paracompactness, stratifiability, metrizability, metacompactness*} $\mathcal{A} = \{A_i \mid 1 \leq i \leq n\}$ *and* $(X, \tau(A_i))$ *has property* P *for* $1 \leq i \leq n$, *then* $(X, \tau(\mathcal{A}))$ *has property* P.

Theorem 3.15 [46]. *If* (X,τ) *is* T_2 *and has property* $P \in$ {*regularity, complete regularity*}, *and* $\mathcal{A} = \{A_\beta \mid \beta \in \mathcal{B}\}$ *such that* $(X, \tau(A_\beta))$ *has property* P *for each* $\beta \in \mathcal{B}$, *then* $(X, \tau(A))$ *has property* P.

Example 3.1 [46]. The preceding does not hold for normality since the Sorgenfrey line is the expansion of the usual topology of the reals by the family of half open intervals.

Theorem 3.16 [7]. *If* $P \in$ {*metrizability, perfect normality, perfectly paracompact, regular-hereditary Lindelöf, stratifiable*}, (X,τ) *is* T_1, $\mathcal{A} = \{A_n \mid n \in N\}$ *such that* $(X, \tau(A_n))$ *is regular for each* $n \in N$, *then* $(X, \tau(\mathcal{A}))$ *has property* P.

Definition 3.3. A collection $\mathcal{J} = \{A_\alpha \mid \alpha \in \mathcal{A}\}$ is R-*open* in (X,τ) if and only if for sets F, G, H which are intersections of a finite number of members of $\mathcal{J}$ and for each $x \in F$ there is a neighborhood $V(x)$ of x and a G containing x such that for each $y \in V(x) \cap (cl_\tau G - F)$ there is H containing y such that $y \notin cl_\tau(H \cap G)$.

Theorem 3.17 [28]. *(a) If* $\mathcal{J} = \{A_\alpha \mid \alpha \in \mathcal{A}\}$ *is a family*

of subsets of X *and* (X,τ) *is regular, then* $(X, \tau(\mathcal{J}))$ *is regular if and only if* $\mathcal{J}$ *is* R*-open in* (X,τ).

(b) If (X,τ) *is metrizable and* $\mathcal{J} = \{A_\alpha | \alpha \in A\}$ *is a* σ*-locally finite collection, then* $(X, \tau(\mathcal{J}))$ *is metrizable if and only if* $\mathcal{J}$ *is* R*-open in* (X,τ).

Theorem 3.18 [46]. *If* (X,τ) *is Hausdorff and has property* P $\in$ {*paracompact, normal metacompact, metrizable, stratifiable*} *and* $\mathcal{J} = \{A_\alpha | \alpha \in A\}$ *such that* $(X, \tau(A_\alpha))$ *has property* P, *then* $(X, \tau(\mathcal{J}))$ *has property* P.

The preceding result holds for a property which is

(i) preserved by finite expansions

(ii) weakly hereditary

(iii) dominated by a locally finite collection of closed sets, and

(iv) implies paracompactness [34].

Theorem 3.19 [45, 24, 27]. *If* (X,τ) *is connected and* $\mathcal{J} = \{A_\alpha | \alpha \in A\}$ *is a collection of subsets of* X *such that finite intersections of members of* $\mathcal{J}$ *are dense, then* $(X, \tau(\mathcal{J}))$ *is connected.*

4. General Results

In this section we discuss results which apply to certain classes of topological properties. In later sections we shall discuss specific results for many specific properties. R shall denote a topological property.

Theorem 4.1 [11]. *A topological space* (X,τ) *is maximal* R *if and only if every continuous bijection from a space* (Y,τ') *with property* R *to* (X,τ) *is a homeomorphism.*

Definition 4.1. A topological property R is *contractive* if (X,τ) has property R and $\tau' \subseteq \tau$ then (X,τ') has property R.

Definition 4.2. A topological property R is (*open, closed, regular closed, point*) *hereditary* if all (open, closed, regular closed, one point) subsets of a space with property R also have property R.

Definition 4.3. A topological property R is:

(1) *open expansive* if (X,τ) has property R and $A \subseteq X$ has property R then $(X, \tau(A))$ has property R.

(2) *closed expansive* if (X,τ) has property R and $A \subseteq X$ has property R then $(X, \tau(X-A))$ has property R.

The covering axioms of compactness, countable compactness and Lindelöf as well as Bolzano-Weierstrass compactness, sequential compactness and connectedness are all contractive, closed hereditary and closed expansive and point hereditary.

Definition 4.4. The graph of $f: (X,\tau) \to (Y,\mathcal{U})$ is an R-*graph* if the graph has property R as a subspace of $(X\times Y, \tau\times\mathcal{U})$.

Theorem 4.2. *Let property* R *be contractive, closed hereditary, and closed expansive. Then the following are equivalent for a topological space* (X,τ) *with property* R:

(1) (X,τ) *is maximal* R;

(2) [11, 30] *the subsets with property* R *are precisely the closed subsets;*

(3) [30] *any continuous surjection from an* R-*space to* X

is a closed quotient map;

(4) [30] *any function with a* R*-graph from* X *is continuous;*

(5) [30] *any function with an* R*-graph into* (X,τ) *is closed.*

Theorem 4.3 [11]. *(a) A topological property* R *is contractive if and only if it is preserved under continuous bijections.*

(b) If R *is a contractive topological property, then* (X,τ) *is maximal* R *if and only if for* $A \notin \tau$, $(X, \tau(A))$ *does not have property* R.

(c) If property R *is contractive, closed and point hereditary, and is closed expansive, and the maximal* R *spaces are* T_1.

(d) If property R *is contractive, closed hereditary and closed expansive, then in maximal spaces the* R *subsets are maximal* R *in their relative topologies.*

(e) Let property R *be contractive, closed and point hereditary and closed expansive. If* $(\Pi_\beta X_\beta, \Pi_\beta T_\beta)$ *is maximal* R, *then* (X_β,τ_β) *is maximal for each* $\beta \in \mathcal{B}$.

(f) [16] *If property* R *is contractive and productive and* $(\Pi_\mathcal{A} X_\partial, \Pi_\mathcal{A} \tau_\partial)$ *is maximal* R, *then* $(X_\partial,\tau_\partial)$ *is maximal* R *for* $\partial \in \mathcal{A}$.

(g) Let property R *be contractive, closed hereditary and closed expansive. If* (X,τ) *is* R, *then* $(X\times X,\tau\times\tau)$ *is maximal* R *only if* (X,τ) *is Hausdorff.*

Theorem 4.4 [11]. *(a) Let property* R *be contractive,*

closed hereditary and closed expansive. If (X,τ) *is maximal* R, $X = \cup\{A_\beta : \beta \in \mathcal{B}\}$, *and* $(A_\beta, \tau/A_\beta)$ *has property* R *for each* $\beta \in \mathcal{B}$ *then* A_β *is* τ *closed and* $(A_\beta, \tau/A_\beta)$ *is maximal* R *for each* $\beta \in \mathcal{B}$.

(b) Let R *be contractive, closed hereditary and closed expansive, and satisfy the condition that a space which is the finite union of subspaces with property* R *also has property* R. *If* $\mathcal{B}$ *is a finite set,* $X = \cup\{A_\beta \mid \beta \in \mathcal{B}\}$, *and* $(A_\beta, \tau|A_\beta)$ *has property* R *for each* $\beta \in \mathcal{B}$, *then* (X,τ) *is maximal* R *if and only if* A_β *is* τ *closed and* $(A_\beta, \tau|A_\beta)$ *is maximal* R *for each* $\beta \in \mathcal{B}$.

If R is a contractive property and (X,τ) is strongly R, then (X,τ) has property R.

Theorem 4.5 [11]. *(a) Let property* R *be contractive, closed and point hereditary and closed expansive. A space* (X,τ) *with property* R *is strongly* R *if* $(X, \tau \vee \mathcal{J})$ *is strongly* R *where* $\mathcal{J}$ *is the topology of finite complements.*

(b) Let property R *be contractive, closed hereditary and closed expansive. Strongly* R *is closed hereditary.*

(c) Let property R *be contractive, closed hereditary and closed expansive. If the product of* T_1 *spaces is strongly* R, *then each coordinate space is strongly* R.

(d) Let property R *be contractive, closed hereditary and closed expansive. If some infinite product of* T_1 *spaces of more than one point is strongly* R, *then the Cantor set is strongly* R.

Definition 4.5. The *dispersion character* $\Delta\tau$ of a topological space (X,τ) is the least cardinal number of a nonempty open set.

Definition 4.6. A topological space (X,τ) is κ-*maximal* if $\Delta\tau \geq \kappa$ and whenever $\tau' \supset \tau$, $\Delta\tau' < \kappa$.

Theorem 4.6 [27]. *(a) The dispersion character of a* T_0-*space is* 1 *or an infinite cardinal.*

(b) If κ *is an infinite cardinal number, or* 1, *then every* T_0 *topological space* (X,τ) *with* $\Delta\tau \geq 1$ *is strongly* κ-*maximal; moreover the finer* κ-*maximal space is* T_1.

Definition 4.7. A topological space (X,τ) is *submaximal* if every dense subset is open.

Theorem 4.7 [32]. *If* $\Delta\tau \geq \kappa$ *where* κ *is an infinite cardinal, then the following are equivalent:*

(1) (X,τ) *is* κ-*maximal;*

(2) (X,τ) *is extremally disconnected and submaximal;*

(3) Every dense-in-itself set in (X,τ) *is open;*

(4) Every topology an X, *which is strictly finer than has isolated points (i.e.,* (X,τ) *is* 2-*maximal).*

Theorem 4.8 [32]. *Let* (X,τ) *be a topological space with infinite dispersion character. Then there exists a* $\Delta\tau$-*maximal topology* τ' *for* X *which is stronger than* τ *and such that* $cl_\tau S = cl_\tau int_\tau cl_\tau S$ *whenever* S *is* τ'-*open.*

Theorem 4.9 [17]. *If property* R *is open (closed) expansive then* (X,τ) *is maximal hereditary* R *with* $\Delta\tau$ *if and*

only if (X,τ) *is* $\Delta\tau$*-maximal.*

Definition 4.8. If R is a topological property such that (X,τ) has R whenever a dense subset has R, then R is *contagious.*

κ-dense, Sanin, Caliber κ, and the countable chain condition are contagious, open hereditary, and open expansive. Second countable is not contagious.

Definition 4.9. For a topological property R, we shall say that (X,τ) is Δ*-maximal* R if (X,τ) has property R and for $\tau' > \tau$ such that $\Delta\tau' = \Delta\tau$, (X,τ) does not have property R.

Theorem 4.10 [14]. *If property* R *is contractive, open hereditary, and open expandable,*

(a) then (X,τ) *is* Δ*-maximal* R *if and only if* (X,τ) *has property* R *and for all* $G \subseteq X$, $G \notin \tau$ *such that* $(G, \tau|G)$ *has property* R, $\Delta(\tau|G) < \Delta\tau$.

(b) then Δ*-maximal* R *is open hereditary.*

(c) and contagious then a Δ*-maximal* R *space is extremally disconnected.*

(d) and contagious then a Δ*-maximal* R *space* (X,τ) *is connected if and only if every open set is dense in* X.

Definition 4.10. A topological space (X,τ) is *semi-regular* if the regular open sets are a base for τ. The *semi-regularization* τ_s of a topology τ is the topology which has the regular open sets of τ as its base. A topological

property R is *semi-regular* if (X,τ) has property R if and only if (X,τ_s) has property R. If (X,τ) is maximal with respect to the property that if $\tau' \supset \tau$ then $\tau'_s \neq \tau_s$, then (X,τ) is *submaximal*.

Connectedness, pseudocompactness, H-closed, QHC, and lightly compact are semi-regular.

Theorem 4.11 [16]. *If property* R *is semi-regular then a maximal* R *topology is submaximal.*

Theorem 4.12 [16]. *If property* R *is contractive, semi-regular, regular closed hereditary, contagious and preserved by finite unions*

(a) then a submaximal space (X,τ) *with property* R *is maximal* R *if and only if for any* $A \subseteq X$ *such that both* $X - int\ A$ *and* A *have property* R, *then* A *is closed.*

(b) then a submaximal space in which every subspace with property R *is closed is maximal* R.

(c) then maximal R *spaces are* T_1 *if* R *is point hereditary.*

5. Compactness

The examples of Hing Tong [54], A. Ramanathan [44], V. K. Balachandran [4] and N. Smythe and C. A. Wilkens [48] showed that maximal compact spaces are not necessarily Hausdorff. Of these we shall just exhibit the Smythe and Wilkens example (Example 5.1a) since there is a finer Hausdorff topology which is minimal (Example 5.1b). Hing Tong's example appears as Example 11.1(a).

Example 5.1. (a) Let $X = \{A,B\} \cup R$ where R is the

real numbers. The topology τ on X is determined as follows: $x \in R$ has the usual neighborhood base; the neighborhood base for A consists of all sets of the form

$$U_n = \{A\} \cup \{(2r - 1, 2r) \mid |r| \geq n, r \text{ an integer}\}$$

for some positive integer n;

the neighborhood base for B consists of all sets of the form

$$V_{(n,d_r)} = \{B\} \cup \{(2r - d_r, 2r + 1 + d_r) \mid |r| \geq n, r \text{ is an integer}\}$$

for some positive integer n and $d_r > 0$, d_r varies with each r and with each n.

(b) Let $X = \{A,B\} \cup R$ with the neighborhood bases for A and the points in R the same as in Example 5.1a. The neighborhood base for the point B consists of all sets of the form

$$V_n = \{B\} \cup \{(2r, 2r + 1): |r| \geq n, r \text{ an integer}\}$$

for some positive integer n.

The one point compactification of the rationals with the usual topology is maximal compact but not Hausdorff since the rationals are not locally compact [37]. Hausdorff quotients of compact spaces are maximal compact, but the following example shows that T_1 quotients of maximal compact spaces are not necessarily maximal compact.

Example 5.2 [11]. Let (Y_i,τ_i) be the one point compactification of N with the discrete topology and y_i the added point for i = 1, 2. Let (X,τ) be the free union of (Y_1,τ_1) and (Y_2,τ_2). Then (X,τ) is maximal compact. Let $X^* = \{y_1,y_2\} \cup N$ and τ^* the topology on X^* having as a base

of open sets $\{n\}$ for $n \in N$, $\{y_1\} \cup (N - A)$ and $\{y_2\} \cup (N - B)$ where A and B are arbitrary finite subsets of N. This space is T_1 and is not maximal compact since $\{y_1\} \cup N$ is compact but is not closed.

6. Countable Compactness

Theorem 6.1 [11]. *(a) A countably compact space* (X,τ) *is maximal countable compact if and only if for every* $G \notin \tau$, *there is a sequence* $S \subseteq X - G$ *with no adherent points in* $X - G$.

(b) For countable spaces, maximal countable compactness is equivalent to maximal compactness.

(c) In a maximal countably compact space every convergent sequence has a unique limit.

Definition 6.1. A topological space is an E_o *space* if every point is a G_δ. A topological space is an E_1 *space* if every point is the intersection of a countable number of closed neighborhoods.

Theorem 6.2 [3]. *(a)* E_1 *spaces are Hausdorff.*

(b) First countable Hausdorff spaces are E_1.

Theorem 6.3. *(a)* [3] *Every countably compact* E_1 *space is maximal countably compact and minimal* E_1.

(b) [3] *In an* E_1 *space, all the countable compact subsets are closed.*

(c) [11] *The countable product of countably compact* E_1 *spaces is maximal countably compact.*

Example 6.1 [11]. (a) The ordinals less than the

first uncountable ordinal Ω with the usual topology is countably compact and E_1 and is therefore maximal countably compact and minimal E_1 and is not compact.

(b) The ordinals less than or equal to the first uncountable ordinal with the usual topology is a Hausdorff compact space and is therefore maximal compact. It is a countably compact space but is not maximal countably compact. The simple expansion of this topology by $\{\Omega\}$ is maximal countably compact and is not compact.

(c) If (X,τ) is a Hausdorff completely regular space, then the Stone-Cech compactification βX is maximal compact but is not necessarily maximal countably compact; for example, βN where N is the natural numbers with the usual topology.

(d) Theorem 6.3c does not extend to arbitrary products since βN is embeddable as a closed subset of products of $[0,1]$ with the usual topology and βN is not maximal countably compact.

Definition 6.2. Let (X,τ) be a noncountably compact space; $X^* = X \cup \{x_o\}$, $x_o \notin X$; $\tau^* = \tau \cup \{G \subseteq X^*: x_o \in G$ and $X - G$ is a closed countably compact subset in $(X,\tau)\}$. (X^*,τ^*) is the *one point countable compactification of* (X,τ).

Definition 6.3. In keeping with the literature, we may say that a topological space is *locally countably compact* if

(i) Every point has a countably compact neighborhood; or

(ii) Every point has a neighborhood whose closure is countably compact; or

(iii) Every neighborhood of a point contains a neighborhood whose closure is countably compact.

A space satisfying (iii) also satisfies (ii); a space satisfying (ii) also satisfies (i); and an E_1 space satisfying (i) also satisfies (iii).

Theorem 6.4 [11]. *The one point countable compactification of a locally countably compact* E_1 *space is maximal countably compact but is not necessarily* E_0.

7. Bolzano-Weierstrass Compactness

Theorem 7.1 [11]. *(a) A space is maximal Bolzano-Weierstrass compact if and only if it is maximal countably compact.*

(b) A space is strongly Bolzano-Weierstrass compact if and only if it is strongly countably compact.

(c) A strongly Bolzano-Weierstrass space is countably compact.

(d) All Bolzano-Weierstrass compact spaces are not strongly Bolzano-Weierstrass compact since there are Bolzano-Weierstrass compact spaces which are not countably compact.

8. Sequential Compactness

Theorem 8.1 [11]. *(a) In a maximal sequentially compact space convergent sequences have unique limits.*

(b) A sequentially compact space is maximal sequentially compact if and only if it is maximal countably compact.

(c) In an E_1 *space maximal countable compactness is*

equivalent to maximal sequential compactness.

(d) An E_1 *space is strongly countably compact if it is strongly sequentially compact.*

(e) The countably compact L^* *spaces* [13, 2, 29] *are maximal sequentially compact.*

Example 8.1 [11]. An E_0 maximal sequentially compact space which is not E_1 and is not an L^*-space.

Let $X = \{a,b\} \cup \{(n,0): n \in N\} \cup \{(n,m): n, m \in N\}$. The topology τ on X is generated by the following neighborhood bases:

$U((n,m)) = \{(n,m)\}$, $n, m \in N$;

$U_r((n,0)) = \{(n,0)\} \cup \{(n,m): m \geq r\}$, $r \in N$;

$U_r(a) = \{a\} \cup \{(n,m): n \geq r, m \in N\}$, $r \in N$;

$U_r(b) = \{b\} \cup \{(n,m): n \geq r, m \geq a_n\} \cup \{(n,0): n \geq r\}$

where a_n varies with each r and for each n in the particular neighborhood, $r \in N$.

9. Lindelöf

Definition 9.1. A topological space (X,τ) is a *quasi-P-space* if the space is T_1 and every G_δ is open. A topological space (X,τ) is a *P-space* if it is quasi-P and completely regular.

Theorem 9.1 [11]. *(a) Every maximal Lindelöf space is a quasi-P-space.*

(b) Every Hausdorff maximal Lindelöf space is a normal P-space.

(c) In a Hausdorff quasi-P-space, Lindelöf subsets are closed.

(d) *Every Lindelöf Hausdorff quasi-*P*-space is maximal Lindelöf and minimal Hausdorff quasi-*P.

(e) *Every Lindelöf* P*-space is maximal Lindelöf and minimal* P.

(f) *A maximal Lindelöf space is first countable, second countable, or separable if and only if it is countable.*

(g) *A maximal Lindelöf space* (X,τ) *is compact, countably compact, or sequentially compact if and only if* X *is finite.*

(h) *Finite products of maximal Lindelöf Hausdorff spaces (minimal* P*-spaces) are maximal Lindelöf (miminal* P*).*

(i) *Let* (X,τ) *be a quasi-*P*-space such that* $X = \cup\{B_i : i \in N\}$ *and* $(B_i, \tau|B_i)$ *is Lindelöf for* $i \in N$. *Then* (X,τ) *is maximal Lindelöf if and only if* B_i *is* τ *closed and* $(B_i, \tau|B_i)$ *is maximal Lindelöf for each* $i \in N$.

(j) *Let* (X,τ) *be a completely regular Hausdorff space such that every free* z*-filter with the countable intersection property is contained in a* z*-ultrafilter with the countable intersection property, then the real compactification* νX *is Lindelöf.*

(k) *If* (X,τ) *is a Lindelöf space which has a cover of* G_δ *sets which has no countable subcover, then* (X,τ) *is not strongly Lindelöf.*

(l) *Let* (X_α,τ_α) *be Lindelöf spaces for which there exist finer maximal Lindelöf Hausdorff topologies,* $\alpha \in A$, *A finite. Then* $(\Pi_A X_\alpha, \Pi_A \tau_\alpha)$ *is strongly Lindelöf.*

Example 9.1 [11]. This is an example of a maximal Lindelöf space which is not Hausdorff and therefore not a P-space.

Let X be the Cartesian plane together with two distinct points a and b. The topology τ on X is defined by the following basic neighborhood systems:

$\{(r,s)\}$ is open for each point (r,s) in the plane;

for the neighborhood systems of a and b, select two disjoint sets A and B such that $A \cup B = R$ and $|A| = |B| = |R|$ where R is the set of real numbers; a neighborhood of a is of the form

$\{a\} \cup \{(r,s): r \in A \text{ and } s \in R - A_r\}$ where A_r is arbitrarily chosen countable subset of R and differs with each $r \in A$ and each neighborhood of a; a neighborhood of b is of the form

$\{b\} \cup \{(r,s): r \in R - (C \cup D)\} \cup \{(r,s): r \in C \text{ and } s \in R - C_r\}$ where C and D are arbitrarily chosen countable subsets of B and A respectively; C_r is an arbitrarily chosen countable subset of R which differs with each $r \in C$; and C, C_r and D differ with each neighborhood of b.

Definition 9.2. In keeping with the literature, we may say that a topological space is *locally Lindelöf* if

(i) Every point has a Lindelöf neighborhood; or

(ii) Every point has a neighborhood whose closure is Lindelöf; or

(iii) Every neighborhood of a point contains a neighborhood whose closure is Lindelöf.

A space which satisfies condition (iii) satisifes

condition (ii), and a space which satisfies condition (ii) satisfies condition (i). A Hausdorff quasi-P-space satisfying condition (i) satisfies (iii).

Definition 9.3. Let (X,τ) be a non-Lindelöf space and $x^* \notin X$. Let $X^* = \{x^*\} \cup X$ and τ^* the topology on X^* such that if $G \in \tau$ then $G \in \tau^*$ and if $x^* \in G \in \tau^*$ then $X^* - G$ is a closed Lindelöf subset of (X,τ). The space (X^*,τ^*) is called the *one point Lindelöf extension* of τ.

Theorem 9.2 [11]. *The one point Lindelöf extension of a locally Lindelöf Hausdorff quasi-*P*-space is a Lindelöf* P*-space and therefore is maximal Lindelöf.*

Example 9.2 [11]. (a) If (X_α,τ_α) is an uncountably infinite set with the discrete topology, the one point Lindelöf extension $(X_\alpha^*,\tau_\alpha^*)$ is maximal Lindelöf. For a cardinal number B, $|A| = B$, then the space (X,τ) which is the free union of $(X_\alpha^*,\tau_\alpha^*)$, $\alpha \in A$ is locally Lindelöf. If $B > \aleph_o$ then the one point Lindelöf extension of (X,τ) is maximal Lindelöf with B non-isolated points and the isolated points are dense.

(b) Assuming the continuum hypothesis, let X be an n_1 set of cardinality $\mathfrak{c}$ and let τ be the order topology. Then (X,τ) is a P-space but is not locally Lindelöf. However, the one point Lindelöf extension of (X,τ) is not Hausdorff but is maximal Lindelöf, and has no isolated points.

10. Connectedness

The topological property which has provoked the most interest in the study of maximal properties is connectedness.

The first work in this field was done by J. P. Thomas in 1968 [52].

Theorem 10.1. *(a)* [45, 24, 43] *Every maximal connected topology is submaximal.*

(b) [24] *Every connected topology can be expanded to a connected submaximal topology.*

Example 10.1 [22]. A connected submaximal space which is not maximal connected.

$X = \{a,b,x,y\}$

$\tau = \{\phi,\{a\},\{b\},\{a,b\},\{a,b,x\}\{a,b,y\},X\}$

Theorem 10.2 [24]. *If* (X,τ) *is a submaximal space with* $\Delta\tau > 1$, *then*

(a) discrete subspaces are closed;

(b) if $|A| < \Delta\tau$, *then* A *is discrete;*

(c) no point has a neighborhood base of cardinality $< \Delta\tau$.

Theorem 10.3 [24]. *(a) No topology is maximal among the connected first countable* T_1 *topologies.*

(b) Every connected subspace of a maximal connected space is maximal connected.

Example 10.2 [52]. (a) For $X \neq \phi$, $x_o \in X$ the space (X,τ) where $\tau = \{\phi\} \cup \{A \subseteq X | x_o \in A\}$ is maximal connected.

(b) For $X \neq \phi$, $x_o \in X$, then the space (X,τ) where $\tau = \{X\} \cup \{A \subseteq X | x_o \notin A\}$ is maximal connected.

(c) For an infinite set X, and $\mathcal{U}$ an ultrafilter finer than $\mathcal{F}$, the filter of finite complements, the

topology $\tau(//)$ is maximal connected and T_1.

Definition 10.1. An *essentially connected space* is a connected space whose connected subsets remain connected as subspaces of every expansion in which X remains connected.

Theorem 10.4 [22]. *(a) Every connected subspace of an essentially connected space is essentially connected.*

(b) Let (X,τ) *be an essentially connected space and let* C *be a connected subset having at least two points. Then* $int_\tau C \neq \phi$ *and if* (X,τ) *is* T_1, *then* $int_\tau C$ *is dense in* C.

Definition 10.2. A space and its topology are called *principal* if each point has a minimum neighborhood, or equivalently, an arbitrary intersection of open sets is open.

Theorem 10.5 [22]. *Apart from the indiscrete doubleton, the concepts of essential and maximal connectedness coincide for principal spaces.*

Definition 10.3. If D is a dense subset of (X,τ), an expansion τ' of τ is D-*maximal* if it is maximal with respect to the properties that $\tau'|D = \tau|D$ and D is τ'-dense in X.

Theorem 10.6 [22]. *If* (X,τ) *is an essentially connected space with a dense maximal connected subspace* D, *then every* D-*maximal expansion of* X *is maximal connected.*

Definition 10.4. A space is *widely connected* if it is connected and every nonsingleton connected subspace is dense.

Theorem 10.7 [22]. *(a) An essentially connected*

T_1-space is widely connected if and only if it has no cut points.

(b) In a maximal connected space, every nonempty non-dense open set has a cut point of X *in its boundary.*

(c) No widely connected Hausdorff space is maximal connected.

(d) No connected Hausdorff space with a dispersion point is strongly connected.

J. A. Guthrie, H. E. Stone, and M. L. Wage [25] have recently used the following method to show the existence of a maximal connected Hausdorff space (Theorem 10.15).

If τ is a connected expansion of the Euclidian topology $\mathcal{E}$ for the real line R, and set $P \subseteq R$ is *pluperfect* (relative to τ) if for every $p \in P$ and each component κ of $R-\{p\}$, $p \in cl_\tau(P\ \kappa)$.

Theorem 10.8 [25]. *If $\tau(P)$ is connected, then P is τ-pluperfect; if $\tau = \mathcal{E}$, this condition is also sufficient.*

A set S is called singular at $X \in S$ (relative to τ) if S is pluperfect and $S-\{X\} \in \tau$. A filter of singluar sets at X will be called a *singular filter* at X, and an expansion σ of τ such that every point has a local base which is a singular filter will be called a *singular expansion* of τ.

Theorem 10.9 [25]. *A connected topology δ is a dense-invariant expansion of τ if and only if there is a connected expansion of δ which is a singular expansion of τ.*

Theorem 10.10 [25]. *Every dense filter expansion of a singular expansion of* $\mathcal{E}$ *is also a singular expansion of a dense-filter expansion of* $\mathcal{E}$, *and conversely.*

Theorem 10.11 [25]. *Every dense-ultrafilter expansion of a connected maximal singular expansion of* $\mathcal{E}$ *is a maximal connected topology.*

Theorem 10.12 [25]. *Suppose* {A,B} *is a* σ*-disconnection of* I = [0,1] *and the set* C(A,B) = I = $(int_{\mathcal{C}}A \cup int_{\mathcal{C}}B)$. *Then* $(C,\mathcal{E})$ *is homeomorphic to the Cantor set.*

Theorem 10.13 [25]. *Let* σ *be a singular expansion of* $\mathcal{E}$. *Each of the following is a sufficient condition for connectedness of* σ.

(a) The expansion is proper at fewer than $\mathcal{C}$ *points.*

(b) Each σ*-neighborhood of each* x *contains an open interval with end point* x.

(c) The space (I,σ) *is normal.*

Theorem 10.14 [25]. *Suppose that all but countably many points have local* σ*-bases which are translates of a given countable singular filter. Then* σ *is connected.*

Theorem 10.15 [25]. *The real line admits a connected maximal singular expansion.*

Theorem 10.16 [25]. *Every Euclidian space is strongly connected.*

I. Baggs [6] has given an example of a countable connected topological space which is not strongly connected.

To describe this space we need the following:

If P is a dense subset of the rationals with the relative topology τ, a subset $G \subseteq P$ is an N-set if $G = \phi$ or if for each $x \in G$ and for every $b > x$, the set $\{y \in G | x < y < b\}$ has nonempty relative interior. A collection of N-sets which is closed under finite intersections is an N-*family*. For each $x \in P$, let $I_x = \{y \in P | x \leq y < x + 1\}$ and let $\mathcal{M}$ be a maximal N-family containing an N-family containing $\{I_x | x \in P\} \cup \{0 \subset P | 0 \in \tau\}$.

If D is a subset of P, let $Q = \{D \subset P | P - D$ is τ-nowhere dense in $P\}$. Then there is a filter $\mathcal{J}$ containing Q such that $\mathcal{J}$ is maximal with respect to the properties

(i) $Q \subset \mathcal{J}$,

and (ii) if $F \in \mathcal{J}$, then F is τ-dense in P.

The topology $\sigma = \mathcal{M} \vee \mathcal{J}$ is maximal perfect (i.e., (P,σ) has no isolated points and any stronger connected topology has an isolated point).

Example 10.3 [6]. Let $\{E_i\}_{i=+\infty}^{-\infty}$ be a countable, disjoint collection of dense subsets of rational numbers indexed by the set of all integers. For each integer n, let $P_n = \{(x,n) | x \in E_n\}$. Let w denote an ideal point. Let $X = \{w\} \cup \{\bigcup_{n=-\infty}^{+\infty} P_n\}$. A neighborhood system for the points of X is as follows:

(a) If n is even, then put a maximal perfect topology σ_n on P_n exactly as described above.

(b) Let $n \neq 1$ be an odd integer and let $p = (x,n) \in P_n$. Then, for each positive integer m, let

$$U_m(p) = \{(y,n+1) \in P_{n+1} \mid |x-y| < 1/m\} \cup \{(y,n-1) \in P_{n-1} \mid |x-y| < 1/m\} \cup \{p\}.$$

(c) Let $n = 1$ and $p = (x,n) \in P_n$. If $x \in (\pi/2,\pi)$, then, for each positive integer m, define $U_m(p)$ as in (b). If $\pi/2 < x < \pi$, then, for each positive integer m, let $U_m(p) = \{(y,n-1) \in P_{n-1} \mid |x-y| < 1/m\} \cup \{p\}$.

(d) If $p = w$, then, for each positive integer m, let $U_m(p) = \{p\} \cup \{(x,n) \mid |n| \geq 2m\}$ be a neighborhood of p.

The fact that this space is not strongly connected is easily shown using 10.7(d).

Definition 10.5. Let a and b be points of a set S and let $H_1,H_2,\cdots,H_n$ be a finite collection of subsets of S. $\{H_1,H_2,\cdots,H_n\}$ is a *simple chain from* a *to* b if

(i) $a \in H_1 - H_2$, $b \in H_n - H_{n-1}$,

and (ii) $H_i \cap H_j \neq \phi$ if and only if $|i-j| \leq 1$, $i = 1,\cdots,n$, $j = 1,\cdots,n$.

Theorem 10.17 [52]. *(a) A space* S *is connected if and only if given any two points* a *and* b *of* S *and any open covering* $\{G_\alpha \mid \alpha \in A\}$ *of* S, *there exists a finite subcollection of* $\{G_\alpha \mid \alpha \in A\}$ *which is a simple chain from* a *to* b.

(b) Let (X,τ) *be a topological space where* X *has at least two elements and is principal. Let* I *be the set of all isolated points of* (X,τ) *and* $J = X - I$. *If* $x \in J$, *let* V_x *be the intersection of all open neighborhoods of* x. *Then a necessary and sufficient condition for* (X,τ) *to be maximal connected is that*

all of the following three statements are true:

(i) $\cup_{x\in J}V_x = X$,

(ii) *If* $x \neq x'$, x *and* $x' \in J$ *then* $V_x \cap V_{x'}$ *has at most one point,*

(iii) *If* $a, b \in (X,\tau)$, *then there exists exactly one simple chain of open sets* V_x *from* a *to* b.

Example 10.4 [52]. Since topologies on finite sets are principal, J. P. Thomas used the preceding result to determine a graphical method for representing maximal connected topologies on finite sets. The members of I are represented by solid dots and the members of J by open dots. V_x is represented by a line segment on which are x and the isolated points in V_x. Below are graphical representations for maximal connected topologies with less than 6 elements.

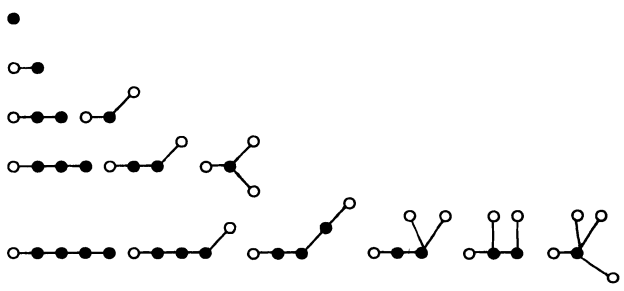
1 element

2 elements

3 elements

4 elements

5 elements

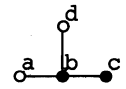


Definition 10.6. A topological space (X,τ) is a *door space* if every subset is either open or closed. A topological space (X,τ) is a *semi-door* space if for $A \subseteq X$ there is $B \in \tau$ such that either $B \subseteq A \subseteq cl_\tau B$ or $B \subseteq X - A \subseteq cl_\tau B$.

We used the method of Example 10.4 to give the following examples.

Example 10.5 [52]. (a) Quotients of maximal connected

connected spaces are not necessarily maximal connected.

For $X = \{a,b,c,d\}$ and τ given by 

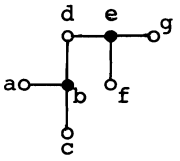
the quotient determined by $R = \{\{a,b\},\{c,d\}\}$ is not maximal.

(b) Products of maximal connected spaces are not necessarily maximal. For $X = \{a,b\}$ and τ given by 

$(X\times X, \tau\times\tau)$ is not maximal connected.

(c) Every maximal connected space is not door or semi-door.

The space of (a) is not door and for $X = \{a,b,c,d,e,f,g\}$, the topology given by



is not semi-door.

Theorem 10.18 [52]. *(a) Every maximal connected topology is* T_o.

(b) Every connected topology on a finite set is strongly connected.

(c) In order that (X,τ) *be maximal connected it is necessary that whenever* $A \subseteq X$ *and* A *is connected and* $X - A$ *is connected, then* $A \in \tau$ *or* $X - A \in \tau$.

Theorem 10.19 [18]. *The number of homeomorphism classes of maximal connected* n*-point topologies is equal to twice the number of* n*-point trees minus the number of* n*-point trees having a symmetry line.*

11. Pseudocompact

Theorem 11.1. *(a)* [43] (X,τ) *is pseudocompact if and only if* (X,τ_s) *is pseudocompact* (τ_s *has as a base the* τ*-regular open sets).*

(b) [43] *Every maximal pseudcompact space is submaximal.*

(c) [10] *A pseudocompact space* (X,τ) *is maximal if and only if for* $G \notin \tau$ *either* G *or* $X - G$ *has a* $\tau(G)$ C*-embeddable copy of the natural numbers* N.

(d) [10] *A pseudocompact space* (X,τ) *is maximal pseudocompact only if for* $G \notin \tau$ *either* G *or* $X - G$ *is not pseudocompact.*

(e) [15] *A maximal pseudocompact space is a* T_1*-space.*

(f) [10] *A countably compact space* (X,τ) *is maximal pseudocompact if and only if for* $G \notin \tau$, $X - G$ *contains a* $\tau(G)$ C*-embeddable copy of* N.

(g) [16] *A* T_1 *completely regular submaximal space is maximal pseudocompact if and only if it is maximal lightly compact.*

(h) [23] *Let* (X,τ) *be pseudocompact, completely regular, first countable and be* $\mathcal{D}$ *be an ultrafilter of* τ*-dense sets. Then* $\tau(\mathcal{D})$ *is maximal pseudocompact.*

Example 11.1. (a) This is Hing Tong's example [48] of a maximal compact space which is not Hausdorff and is also maximal pseudocompact.

Let $X = \{a,b\} \cup N \times N$ where N is the set of natural numbers. The topology τ on X has a base of all sets of the following forms:

$\{(m,n)\}$ for $(m,n) \in N \times N$;

$\{a\} \cup \{(m,n): m = 2k, k \in N, \text{ and } n \in N - A_m\}$;

$\{b\} \cup \{(m,n): m \in N - (A \cup B)\} \cup \{(m,n): m \in B \text{ and } n \in N - B_m\}$

where A and B are arbitrarily chosen finite subsets of the even and odd positive integers respectively, and A_m and B_m are arbitrarily chosen finite subsets of N and vary with each permissible m.

(b) A pseudocompact T_D space which is not strongly pseudocompact.

Let X be infinite, $x_o \in X$. The topology τ on X consists of all sets $A \subseteq X$ such that $x_o \in A$.

(c) A maximal countably compact space which is not maximal pseudocompact.

Let X = [0,1] and let τ be the usual topology.

(d) This example is an example of P. Urysohn [56] and is an example of a maximal pseudocompact space which is not maximal countably compact.

Let $X = \{a,b\} \cup N \times I$ where I is the set of integers. The neighborhood bases for τ consist of the following sets:

$\{(m,n)\}$ for $m \in N$, $n \in I - \{0\}$;

$U_r((m,0)) = \{(m,0)\} \cup \{(m,n): |n| > r\}$ for $r \in N$;

$U_r(a) = \{a\} \cup \{(m,n): m \geq r, n > 0\}$ for $r \in N$;

$U_r(b) = \{b\} \cup \{(m,n): m \geq r, n < 0\}$ for $r \in N$.

12. Lightly Compact (Feebly Compact)

Definition 12.1. A space (X,τ) is *lightly compact* (feebly compact) if every locally finite family of non-void open sets is necessarily finite.

Theorem 12.1 [49]. *On a space* (X,τ), *the following*

are equivalent:

(a) (X,τ) *is lightly compact*

(b) If $\mathcal{U}$ *is a countable open cover of* X*, then there exists a finite subcollection of* $\mathcal{U}$ *whose closures cover* X*;*

(c) Every countable open filter base on X *has an adherent point. Lightly compact is semiregular, contractive regular closed hereditary, contagious and preserved by finite unions.*

Theorem 12.2 [43]. *Every maximal lightly compact space is submaximal.*

Example 12.1 [43]. A light compact space which is submaximal but not maximal.

Let X be infinite $x_o \in X$ and $\tau = \{\phi\} \cup \{V \subseteq X | x_o \in V\}$.

Example 11.1 (b) is a lightly compact space which is not strongly lightly compact.

Theorem 12.3 [16]. *A* T_1 *completely regular submaximal space is maximal lightly compact if and only if it is maximal pseudocompact.*

13. QHC and H-Closed

Definition 13.1. A topological space (X,τ) is QHC (*quasi-H-closed*) if every open filter base has a cluster point or equivalently if every open cover has a finite subfamily whose closures cover X. In a Hausdorff space, QHC is called H-*closed* and is equivalent to being closed in any Hausdorff space in which it may be embedded.

QHC is semiregular, contractive, regular closed hereditary, contagious and preserved by finite unions.

Every H-closed space is strongly H-closed, but Example 11.1 (b) is a QHC space which is not strongly QHC. Example 11.1 (a) is a maximal QHC space which is not Hausdorff and Example 11.1 (d) is a maximal H-closed space.

Definition 13.2. A subset B of a topological space (X,τ) is *interiorly* QHC if every open cover of B has a finite subfamily whose closures cover int A.

Theorem 13.1 [13]. *A* QHC*-space is maximal* QHC *if and only if it is submaximal and if, for any* $A \subseteq X$, A *is interiorly* QHC *and* X*-int* A *is* QHC, *then* A *is closed.*

14. K-Dense Spaces

Definition 14.1. For an infinite cardinal κ a topological space (X,τ) is κ*-dense* if it has a dense subset of cardinality $\leq\kappa$ ($\aleph_0$-dense is separable).

Definition 14.2. A κ-dense space (X,τ) is Δ-maximal κ-dense if for $\tau' \supset \tau$, (X,τ') is not κ-dense if $\Delta\tau' = \Delta\tau$.

κ-dense is open hereditary, open expandable, contagious and contractive.

Theorem 14.1 [14]. *(a) If* (X,τ) *is* κ*-dense with* $\Delta\tau > \kappa$ *then* (X,τ) *is* Δ*-strongly* κ*-dense. Furthermore if* D *is a* κ*-dense set then there is a finer* Δ*-maximal* κ*-dense topology* τ_D^* *such that* $\Delta\tau_D^* = \Delta\tau$, D *is dense in* τ_D^* *and if* C *is a* κ*-dense set, then* $C \cap D \neq \phi$.

(b) The reals are Δ-strongly separable.

(c) If (X,τ) is Δ-maximal κ-dense with $\Delta\tau \geq \kappa$, and B is κ-dense such that $\Delta(\tau|B) = \kappa$, then $(B, \tau|B)$ is Δ-maximal κ-dense.

(d) The rationals are Δ-strongly separable.

(e) If (X,τ) is κ-dense with $\Delta\tau \geq \kappa$ and D is a dense subset, $|D| \leq \kappa$, $\Delta(\tau|D) = \kappa$ then (X,τ) is Δ-strongly T_1 κ-dense.

(f) If (X,τ) is Δ-maximal κ-dense with $\Delta\tau > \kappa$, then (X,τ) is resolvable.

(g) If (X,τ) is Δ-maximal κ-dense, D dense, $|D| \leq \kappa$ and $\Delta(\tau|D) = \kappa$, then (X,τ) is κ-maximal.

15. Countable Chain Condition

Definition 15.1. A topological space (X,τ) has the *countable chain condition* (CCC) if every collection of disjoint open sets is countable.

CCC is contagious, open hereditary, contractive and open expandable.

Definition 15.2. A topological space (X,τ) with CCC is *Δ-maximal* CCC if for $\tau' \supset \tau$, $\Delta\tau' = \Delta\tau$, then (X,τ') is not CCC.

Theorem 15.1 [14]. *(a) Every dense subset of a* CCC *space is* CCC.

(b) A CCC *space (X,τ) is Δ-maximal* CCC *if and only if it is $\Delta\tau$-maximal.*

(c) Every CCC *space is Δ-strongly* CCC.

16. Heriditary Properties

Definition 16.1. For a topological property R, a space (X,τ) is Δ-*maximal hereditary* R if for any $\tau' \supset \tau$, either (X,τ') is not hereditary R or $\Delta\tau' < \Delta\tau$.

Theorem 16.1 [17]. *If* R *is open (closed) expandable, then* (X,τ) *is* Δ*-maximal hereditary* R *if and only if* τ *is* $\Delta\tau$*-maximal.*

Theorem 16.2 [17]. *A* T_o*-space* (X,τ) *is strongly hereditary*

(1) κ*-dense if and only if* $|X| \leq \kappa$.

(2) $[a,b]$*-compact if and only if* $|X| \leq a$.

Theorem 16.3 [14]. *A second countable space* (X,τ) *is strongly second countable only if* $|X| \leq \aleph_o$.

17. Non T_i-Spaces

Although it is generally accepted policy to study maximal topologies for some property R, J. P. Thomas [51] chose to study topological spaces which do not have property R but for which any stronger topology has property R. The following results are his.

Theorem 17.1. *(a) A topological space* (X,τ) *is maximal non-Kolmogoroff (non-*T_0*) if and only if for some* $a, b \in X$, τ *has as a base* $\{\{a,b\}\} \cup \{\{x\} | x \in X, x \neq a,b\}$.

(b) A topological space (X,τ) *is maximal non-accessible (non-*T_1*) if and only if* τ *has as its base for some* $a, b \in X$, $\{\{a,b\}\} \cup \{\{x\} | x \in X - \{a\}\}$.

(c) Any non-Kolmogoroff (non-accessible) space is

strongly non-Kolmogoroff (non-accessible).

(d) *If* (X,τ) *is non-separated (non-Hausdorff) then there exists a maximal non-separated and maximal non-regular topology* τ' *stronger than* τ.

(e) *If* (X,τ) *is separated and non-regular then there is a strictly finer topology* τ' *for which* (X,τ') *is separated and non-regular and thus there do not exist Hausdorff maximal non-regular spaces and the maximal non-Hausdorff spaces are the same as the maximal non-regular spaces.*

(f) *The product of two non-trivial topological spaces is not maximal non-Kolmogorff, maximal non-accessible, nor maximal non-separated.*

18. Door Spaces

In this section all spaces considered are Hausdorff spaces of infinite cardinality.

Definition 18.1. A *door space* is a space in which every subset is either open or closed.

Definition 18.2. A nondiscrete door space is *maximal door* if the only finer door topology for the set is discrete.

Theorem 18.1 [31]. *A Hausdorff space* (X,τ) *is a non-discrete door space if and only if* $X = S \cup \{p\}$ *where* S *is an infinite discrete set and* p *is a point such that the restriction of its neighborhoods to* S *forms a filter on* S.

Definition 18.3. For an infinite cardinal m, a subset S of βm (m discrete) is *strongly discrete* if for each $s \in S$

there is a neighborhood $U_s \subset \beta m$ of s such that if $s \neq t$, then $U_s \cap U_t \cap m = \emptyset$.

Theorem 18.2. *For an infinite door space* (X,τ), *the following are equivalent:*

(a) (X,τ) *is maximal door;*

(b) [31] $X = S \cup \{p\}$ *where* S *is an infinite discrete set and* p *is a point such that the restrictions of its neighborhoods to* S *forms an ultrafilter in* S;

(c) [21] (X,τ) *can be embedded in some* βm *in such a way that* S *is strongly discrete.*

Theorem 18.3 [21]. *Every countable nondiscrete door space which can be embedded in* βm *is a maximal door space.*

Theorem 18.4 [31]. *For every maximal door space* (X,τ) *there is a discrete space* m *such that* X *can be embedded in* βm; *furthermore* m *may be taken as* $|X|$.

Theorem 18.5 [21]. *For every infinite cardinal* $\mathfrak{m}$, *there exists a nondiscrete door space* X *with* $|X| > m$ *such that* X *can be embedded in* βm *but* X *is not maximal door. In particular, there is a nondiscrete door space of cardinality* $2^{\aleph_0}$ *which is not maximal door, but can be embedded in* $\beta\aleph_0$.

19. Functionally Maximal

All spaces to be considered in this section are T_1.

Definition 19.1. For a class of topological spaces Σ we define an expansion σ of τ to be *functionally invariant with respect to* Σ (Σ-*invariant*) if for each space $Y \in \Sigma$ the σ-continuous functions into Y are all τ-continuous.

A topology τ is called Σ-maximal or functionally maximal with respect to Σ if no proper expansion of τ is Σ invariant.

If Σ is a singleton $\{Y\}$, we write Y-invariant instead of $\{Y\}$-invariant.

A space is Σ-regular if X can be embedded in a product of spaces from Σ. The class of Σ-regular spaces is denoted Σ^r.

Theorem 19.1 [23]. *Let Σ be a class of spaces and let (X,σ) be an expansion of (X,τ). Then σ is Σ-invariant if and only if σ is Σ^r-invariant.*

Corollary 19.1 [23]. *(a) Let Y be a space and Σ be the class of all spaces having the topology induced by the continuous functions into Y. Then an expansion is Σ-invariant if and only if it is Y-invariant.*

(b) An expansion is Tychonoff-invariant if and only if it is R-invariant.

(c) An expansion is functionally invariant with respect to the class of all zero-dimensional Hausdorff spaces if and only if it is functionally invariant with respect to the discrete doubleton.

(d) An expansion σ of τ is functionally invariant with respect to all topological spaces if and only if it is functionally invariant with respect to the Sierpinski doubleton (if and only if $\sigma = \tau$).

Definition 19.2. A space (X,τ) is *perfectly Hausdorff* if every point is a zero set.

Theorem 19.2 [23]. *Let (X,τ) be a perfectly Hausdorff*

space and let σ *be a pseudocompact expansion of* τ. *Then* σ *is an* R-*invariant expansion of* τ.

Theorem 19.3 [23]. *Let* (X,τ) *be a first countable Tychonoff space and suppose* σ *is an* R-*invariant expansion of* τ. *Then* $\tau \leq \sigma$ ($\tau \leq \sigma$ means that $\tau \subseteq \sigma$ and $cl_\tau V = cl_\sigma V$ for every $V \in \tau$).

Theorem 19.4 [23]. *There exist connected expansions of the usual topology on* R *which are not* R-*invariant.*

Definition 19.3. If σ is an expansion of τ, $x \in X$ is an *improper point of the expansion* if σ and τ determine the same neighborhood system at x (neighborhoods need not be open). If the set of improper points is τ-dense, the expansion is said to be improper on a dense set.

Corollary 19.2 [23]. *If* $\sigma \geq \tau$ *is improper on a dense set, then* σ *is an* R-*invariant expansion.*

Definition 19.4. S is *locally dense relative to* τ if $S = V \cap D$ where $V \in \tau$ and D is τ-dense.

Definition 19.5. (X,τ) is *essentially connected* if every connected expansion has the same connected subsets.

Theorem 19.5 [23]. *Let* (X,τ) *be essentially connected and locally connected, and let* $\mathcal{D}$ *be a pairwise disjoint family of locally dense sets. Let* σ *be the expansion whose subbase is* $\tau \cup \mathcal{D}$. *The following are equivalent:*

(a) σ *is* R-*invariant,*

(b) σ *is connected,*

(c) σ is improper on a dense set.

Theorem 19.6 [23]. *A regular-maximal space is submaximal.*

Theorem 19.7 [23]. *Let* (X,τ) *be completely regular, first countable space and* $\mathcal{D}$ *be an ultrafilter of* τ*-dense sets. Then* $\tau(\mathcal{D})$ *is* R*-maximal.*

20. Maximum and Minimum Spaces

Closely allied with the concept of maximal and minimal spaces is that of maximum and minimum spaces. The best known example is the minimum T_1-topology which is the topology of finite complements. R. E. Larson [34] has characterized the maximum and minimum topological spaces and the results given here are his.

Definition 20.1. A topological (X,τ) is *completely homogeneous* if every one-to-one mapping of X onto itself is a homeomorphism.

Theorem 20.1. *(a) Any subspace of a completely homogeneous space is completely homogeneous.*

(b) If (X,τ) *is non-discrete and completely homogeneous, then* (X,τ) *is connected and not Hausdorff.*

(c) The set of completely homogeneous topologies on a set X *form a linearly ordered sublattice of the lattice of all topologies on* X.

(d) The only completely homogeneous topologies on X *are*

(i) the indiscrete topology

(ii) the discrete topology

(iii) topologies of the form

$$\tau = \{G \subseteq X \mid |X-G| < m\} \cup \{\phi\} \text{ where } \aleph_0 \leq m \leq |X|.$$

(e) If X *is a set of cardinality* m *and* n *is the cardinality of the set of all infinite cardinals less than or equal to* m, *then the number of minimum topologies on* X *is* n + 2.

Theorem 20.2. *Given a topological space* (X,τ), *the following are equivalent:*

(a) (X,τ) *is completely homogeneous.*

(b) (X,τ) *is minimum* P *for some topological property* P.

(c) (X,τ) *is maximum* Q *for some topological property* Q.

21. Complementary Properties

A recent object of interest in both minimal and maximal topological spaces is that of complementary properties introduced by R. E. Larson [35] in 1973.

Definition 21.1. A topological property P is *expansive* (*contractive*) if for any topology τ with property P any finer (coarser) topology τ' also has property P.

Definition 21.2. If P is an expansive topological property and Q is a contractive topological property then P and Q are called *complementary* if a topology is minimal P if and only if it is maximal Q.

Definition 21.3. A topological space is *loosely nested* if it is nested and the closed derived sets of points are point closures.

Theorem 21.1 [35]. *(a)* T_0 *and loosely nested are*

complementary.

(b) T_D *and nested are complementary.*

(c) T_1 *and the property that all proper closed sets are finite are complementary.*

Definition 21.4. A principal topological space is *principal of order* n if it possesses at most n distinct minimal open sets.

Theorem 21.2 [35]. *Disconnected and principal of order two are complementary topological properties.*

Definition 21.5. A topological space is *filter-connected* if either the nonempty open sets or nonempty closed sets form a filter base.

Theorem 21.3 [35]. *Door and filter-connected are complementary.*

Definition 21.6. A topological space is H-*compact* if the following conditions hold:

H(i) Every open filter has a cluster point.

H(ii) Every open filter with a unique cluster point converges.

R. E. Larson [35] conjectured that Hausdorff and H-compact are complementary but it has been shown recently [12] that Hing Tong's example [Example 11.1(a)] is maximal H-compact, maximal H(i) and maximal H(ii) and not Hausdorff.

22. Some Unsolved Problems

In this final section we listed some unsolved questions

which have arisen during investigation of the topics discussed in this paper. They are listed by section number for easier reference.

4-1. *What are necessary and sufficient conditions for a space to be strongly* R*?*

5-1. *Are all compact spaces strongly compact?*

6-1. *Under what conditions is* βX *maximal countably compact?*

6-2. *Are all countably compact spaces strongly countably compact?*

8-1. *Are all sequentially compact spaces strongly sequentially compact?*

8-2. *Are there maximal countably compact spaces which are not sequentially compact?*

11-1. *Without reference to any expansions what are necessary and sufficient conditions for a space to be maximal pseudocompact?*

19-1. *What are complementary properties for each property discussed in this paper?*

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