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by

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0. Introduction

One of the most fruitful periods of point set topology was that in which the properties and importance of locally finite collections and discrete collections were established. Central in this work were the theorems of Bing, Michael, Nagata, Smirnov, and Stone. The class of paracompact spaces suddenly became significant. The metrizable spaces were found to be exactly those regular spaces with σ-locally finite bases (or, equivalently, σ-discrete bases).

These beautiful results inspired investigations of many similarly defined classes of spaces. For example we can define 24 classes of spaces by choosing one from each of groups A, B, C: (A: σ, countably, ————), (B: hypo, meta, para, ————), (C: compact, Lindelöf) spaces. We similarly can define 12 more classes: spaces with (D: σ, ————), (E: point, locally, star), (F: finite, countable) bases.

Not all of the above classes are distinct. Not all of the above classes are interesting. But a surprising (considering the silly way we have defined them) number of these classes are important. Further, some of the implications are very powerful tools. With three exceptions, these classes and the implications between them have been well understood for some time. For example, of the twelve base properties, six imply metrizable, and five do not, even adding either paracompactness (example: the Michael line [Mi]) or Moore (example: Heath's V-space [H]).

The three exceptions are the paralindelöf spaces, the σ-paralindelöf spaces, and the spaces with σ-locally countable bases. Because the other properties are well understood, it is frustrating that these do not fall into line. What is it that makes them different and more difficult?

(For completeness, let us recall that a space X is paralindelöf [σ-paralindelöf] if every open cover has an open refinement which is locally countable [the union of countably many locally countable families]).

In our opinion, the most important open question in this area is whether paralindelöf spaces are paracompact. Section 1 records our only progress on that question--an extension of some results of Tall [T]. Paralindelöf spaces are strongly collectionwise Hausdorff, and this enables us to conclude that paralindelöf, first countable, σ-discrete spaces are metrizable. Analogous results with "point" replaced by "closed Lindelöf subset" are presented.

Parallel to Fedorčuk's theorem that paracompact spaces with a σ-locally countable base are metrizable, we show that subparacompact spaces with a σ-locally countable base are Moore. Using a theorem of H. E. White, we show that spaces with a σ-locally countable base have a dense metrizable subspace. Example 2.5 is the first example of a nonmetrizable space with a σ-locally countable base; it is a metacompact Moore space. Example 2.6, due to Gruenhage, is a space with a σ-locally countable base that is not even countably metacompact.

Tall's theorem that countably paracompact, σ-paralindelöf spaces are paralindelöf is the main result of section 3. In section 4 we prove a generalization of the fact that MA + ¬CH gives a Q set. We show that under MA every T_1 space of cardinality less than continuum with a σ-point finite base is perfect and metacompact. The paper concludes with a list of open questions.

At the conference this volume records, we presented our results and asked: 1. Must spaces with a σ-locally countable base be Moore? metacompact? perfect? 2. Is there an honest example of a regular space of cardinality ω_1 with a point countable base that is not perfect? Gary Gruenhage answered question 1 with Example 2.6, which is related to some examples in [DGN]. We wish to thank him for allowing us to include his example in this paper, and for pointing out Theorem 1.6 a) and c). Peter Davies has answered question 2 with an example. We have replaced our original complicated proof of Corollary 2.4 with a simple proof due to H. E. White.

By including Theorem 2.1, due to Fedorčuk, Example 2.6, due to Gruenhage, and Theorem 3.1 due to Tall, we have made this paper a survey article as well as a research report. Everything which we know about paralindelöf spaces and spaces with a σ-locally countable base has been included.

Let us note that the beautiful theorems of Bing, Michael, Nagata, Smirnov, and Stone require regularity in the hypothesis. There is a countable, first countable Hausdorff space that is not paracompact. Therefore, throughout this paper, *all spaces are* T_1 *and regular.*

1. Separation Properties of Paralindelöf Spaces

We begin by defining some separation properties. There is a subtle but significant difference between collectionwise Hausdorff and strongly collectionwise Hausdorff. We do not contend that L-cwr is an intrinsically important property. We have found it convenient because it implies both pseudonormal and strongly collectionwise Hausdorff, and it is exactly what can be concluded by the argument of Theorem 1.7.

1.1. *Definition*. a) A collection $\mathcal{A}$ of subsets of a space X is said to be *discrete* iff every point of X has a neighborhood meeting (i.e. having nonempty intersection with) at most one member of $\mathcal{A}$. We will informally say discrete collection of points when we should say discrete collection of singletons.

b) A space X is said to be *pseudonormal* [*L-regular*] iff every pair of disjoint closed subsets of X, one of which is countable [Lindelöf] can be separated by disjoint open subsets of X.

c) A space X is said to be *collectionwise Hausdorff (cwH)* [*Lindelöf-collectionwise Hausdorff (L-cwH)*] if every discrete collection of points [closed Lindelöf sets] can be separated by a disjoint family of open sets.

d) A space X is said to be *strongly collectionwise Hausdorff (scwH)* [*sL-cwH*] if every discrete collection of points [closed Lindelöf sets] can be separated by a discrete family of open sets.

e) A space X is said to be *cwr* [*L-cwr*] if whenever $\mathcal{Y} = \{Y_i : i \in I\}$ is a closed discrete collection of points [closed Lindelöf sets], and H is a closed set disjoint from $\cup\mathcal{Y}$, then there is a disjoint family $\{U\} \cup \{V_i: i \in I\}$ of open subsets of X with $U \supset H$ and $Y_i \subset V_i$ for all $i \in I$.

f) A space X is said to be *σ-discrete* [*Lσ-discrete*] if $X = \cup\{\cup\mathcal{Y}_n: n \in \omega\}$, where each n is a discrete collection of points [closed Lindelöf sets].

1.2. *Lemma. a) A space* X *is* cwr *iff* X *is* scwH.

b) A space X *is* L-cwr *iff* X *is* L-*regular and* sL-cwH.

Proof. We prove a) only, as b) is parallel.

First, we assume X is cwr. Let $\mathcal{Y}' = \{\{y_i\}: i \in I\}$ be a discrete collection of points. Apply cwr with $\mathcal{Y} = \mathcal{Y}'$, $H = \emptyset$ to get a disjoint open family $\mathcal{V}' = \{V_i': i \in I\}$, with $y_i \in V_i'$. Apply cwr again with $\mathcal{Y} = \mathcal{Y}'$, $H = \overline{\cup\mathcal{V}'} - \cup\mathcal{V}'$ to get a disjoint open family $\mathcal{V}'' = \{V_i'': i \in I\} \cup \{U''\}$ with $y_i \in V_i''$ and $U'' \supset H$. Then $\{V_i' \cap V_i'': i \in I\}$ is a discrete open family separating $\mathcal{Y}$.

Conversely, we assume X is scwH. Let $\mathcal{Y}$ and H be as in the hypothesis of cwr. Let $\mathcal{W}$ be a discrete family of open sets separating $\mathcal{Y}$. By regularity for each i there is an open set S_i satisfying $y_i \in S_i \subset \overline{S}_i \subset X - H$. Set $V_i = W_i \cap S_i$. Then $\{X - \overline{\{V_i: i \in I\}}\} \cup \{V_i: i \in I\}$ satisfies the conclusion of cwr.

1.3. *Corollary. A normal* cwH *space is* scwH. *A normal* L-cwH *space is* sL-cwH.

1.4. *Example*. The Tychonoff plank is cwH but not scwH, nor L-regular.

1.5. *Remarks*. Wage's [W] Lσ-discrete cwH not normal

Moore space is not scwH. Van Douwen and Wage have constructed, assuming P(c), a σ-discrete, cwH not normal Moore space. If it were scwH, then by Theorem 1.6 a), it would be metrizable. The only example known to the authors of scwH nonmetrizable Moore spaces is that in [F]; an extra axiom of set theory, $E(\kappa)$, is needed to construct it.

Another difference between cwH and scwH arises when λ is a singular cardinal of cofinality ω. If a space X is scwH with respect to collections of cardinality less than λ, X is scwH with respect to collections of cardinality λ. This statement is false if scwH is replaced by cwH [F'].

The properties scwH and σ-discrete [sL-cwH and Lσ-discrete] fit together well. Recall that a space is called a σ space if it has a σ-discrete closed network.

1.6. *Theorem. a) A σ-discrete* scwH *space* X *is paracompact.*

b) A first countable σ-discrete scwH *space* X *is metrizable.*

c) An Lσ*-discrete* sL-cwH σ *space* X *is paracompact.*

d) An Lσ*-discrete* sL-cwH *space* X *is collectionwise normal.*

Proof. a) Let $\mathcal{U}$ be an open cover of $X = \cup\{\cup\mathcal{Y}_n: n \in \omega\}$ where each $\mathcal{Y}_n$ is a discrete collection of singletons. Let $\mathcal{V}_n$ separate $\mathcal{Y}_n$. We can modify $\mathcal{V}_n$ so that each $V \in \mathcal{V}_n$ is contained in some $U \in \mathcal{U}$. Then $\cup\{\mathcal{V}_n: n \in \omega\}$ is a σ-discrete refinement of $\mathcal{U}$, demonstrating that X is σ-paracompact. A regular σ-paracompact space is paracompact.

b) Arguing as in a), X has a σ-discrete base.

c) As in a) we show that X is σ-paracompact.

d) Let $\mathcal{Y} = \{Y_i: i \in I\}$ be a closed discrete collection of closed subsets of X. Let $X = \cup\{\mathcal{K}_m: m \in \omega\}$, where each $\mathcal{K}_m$ is a discrete collection of closed, Lindelöf sets. Define for $i \in I$, $m \in \omega$, $\mathcal{Y}_m = \{Y \cap K: K \in \mathcal{K}_m, Y \in \mathcal{Y}\}$. Then $\mathcal{Y}_m$ is a closed discrete collection of closed Lindelöf subsets of X. Because X is sL-cwH, there is a discrete family $\mathcal{V}_m = \{V(Y,K,m): Y \in \mathcal{Y}, K \in \mathcal{K}_m, m \in \omega\}$ of open subsets of X with $Y \cap K \subset V(Y,K,m)$. As $\mathcal{Y}$ is closed, discrete, we may additionally have $V(Y,K,m) \cap (\cup\mathcal{Y}) \subset Y$. Now define $\hat{V}(Y,K,m) = V(Y,K,m) - \overline{\cup\{V(Y',K',n): Y' \neq Y, n \leq m\}}$. $\hat{V}(Y,K,m)$ is open because each $\mathcal{V}_n$ is discrete; $\hat{V}(Y,K,m) \supset Y \cap K$ by the additional requirement; $\hat{V}(Y,K,m)$ meets $\hat{V}(Y',K',m')$ implies $Y = Y'$, as either $m \leq m'$ or $m' \leq m$.

Define $V(Y) = \cup\{\hat{V}(Y,K,m): m < \omega, K \in \mathcal{K}_m\}$. $\mathcal{V} = \{V(Y): Y \in \mathcal{Y}\}$ simultaneously separates $\mathcal{Y}$.

1.7. *Theorem. A paralindelöf space* X *is* L-cwr; *a fortiori* sL-cwH, L-cwH, scwH, cwH, *and pseudonormal.*

From Theorems 1.6 and 1.7 we get

1.8. *Corollary. Let* X *be a paralindelöf space.*

a) If X *is* σ*-discrete,* X *is paracompact.*

b) If X *is first countable and* σ*-discrete,* X *is metrizable.*

c) If X *is a* Lσ*-discrete* σ *space,* X *is paracompact.*

d) If X *is* Lσ*-discrete,* X *is collectionwise normal.*

Proof of Theorem 1.7. Let $\mathcal{K} = \{K_i: i \in I\}$ be a discrete collection of closed Lindelöf subsets of X; let H be a closed subset of X disjoint from $\cup\mathcal{K}$. We will define a series of open covers as indicated by the diagram below. $\mathcal{A} > \mathcal{C}$ means that $\mathcal{C}$ is a refinement of $\mathcal{A}$.

$$\mathcal{A} \underset{\text{reg}}{>} \mathcal{C} \underset{\text{pL}}{>} \mathcal{C}' \underset{\text{wit}}{>} \mathcal{W} \underset{\text{pL}}{>} \mathcal{W}' \underset{\text{wit}}{>} \mathcal{J}$$

Let $\mathcal{A} = \{X - K_i: i \in I\} \cup \{X - H\}$. For $k \in \cup\mathcal{K}$, define $K(k)$ so that $k \in K(k) \in \mathcal{K}$; for $h \in H$ define $K(h)$ to be H. Let $Y = \cup\mathcal{K} \cup H$.

By regularity, for every $y \in Y$ we can find an open set U_y satisfying

$$y \in U_y \subset \overline{U}_y \subset (X - Y) \cup K(y)$$

Then $\mathcal{C} = \{X-Y\} \cup \{U_y: y \in Y\}$ is an open cover of X. Since X is paralindelöf, there is $\mathcal{C}'$, a locally countable open refinement of $\mathcal{C}$.

Let $\mathcal{W}$ be an open cover of X, refining $\mathcal{C}'$ and witnessing that $\mathcal{C}'$ is locally countable (that is, for all $W \in \mathcal{W}$, $\{C \in \mathcal{C}': W \cap C \neq \emptyset\}$ is countable). We repeat the process. Let $\mathcal{W}'$ be a locally countable open cover of X refining $\mathcal{W}$; let $\mathcal{J}$ be an open cover of X refining $\mathcal{W}'$ and witnessing that $\mathcal{W}'$ is locally countable.

Each K_i is a closed Lindelöf subset of X. So, for each $i \in I$, there is $\mathcal{S}_i$, a countable subset of $\mathcal{J}$ such that $\cup\mathcal{S}_i \supset K_i$. We can and do assume that every $S \in \mathcal{S}_i$ meets K_i. Let $\mathcal{S}^* = \{W \in \mathcal{W}': W \cap H \neq \emptyset\}$. Let $\mathcal{S} = \cup\{\mathcal{S}_i: i \in I\} \cup \mathcal{S}^*$.

Define $q: \mathcal{S} \to \{H\} \cup \mathcal{K}$ so that $S \cap q(S) \neq \emptyset$. By our choice of $\mathcal{A}$ and our insistence on refinements, q is unique. We define an equivalence relation $\sim$ on $\mathcal{S}$ by $S \sim S'$ iff there is a finite sequence $S_0, S_1, \cdots, S_n$ satisfying

i) $S_0 = S$, $S_n = S'$

ii) for all $m < n$ $S_m \cap S_{m+1} \neq \emptyset$

iii) for all $m < n$ $q(S_m) \neq q(S_{m+1})$.

We aim at showing that each $\sim$ equivalence class is countable. It is sufficient to show that for each $S \in \mathcal{S}$, $M(S) = \{S' \in \mathcal{S}: S \cap S' \neq \emptyset, q(S) \neq q(S')\}$ is countable. This reduces to several claims.

Claim 0. If $q(S) = H$, then $M(S) \cap \mathcal{S}^* = \emptyset$.

Claim 1. If $q(S) \in K$, then $M(S) \cap \mathcal{S}^*$ is countable. This is easy because $S \in \mathcal{J}$ which witnesses that $\mathcal{W}' \supset \mathcal{S}^*$ is locally countable.

Claim 2. For all $S \in \mathcal{S}$, $M(S) \cap \mathcal{S} - \mathcal{S}^*$ is countable. Each S is contained in some element of $\mathcal{W}$. Thus, each S meets at most countably many elements of $\mathcal{C}'$. Each element of $\mathcal{C}'$ meets at most one element of $\mathcal{K}$. So $M(S) \cap \mathcal{S} - \mathcal{S}^* \subset \cup\{\mathcal{S}_i: \exists C \in \mathcal{C}', S \cap C \neq \emptyset, C \cap K_i \neq \emptyset\}$. Recalling that each $\mathcal{S}_i$ is countable, we demonstrate Claim 2, and that each $\sim$ equivalence class is countable.

Let E be the set of all $\sim$ equivalence classes. Enumerate each $e \in E$ as $\{S(e,m): m < \omega\}$. For each $e \in E$, $m < \omega$, we define $\hat{S}(e,m) = S(e,m) - \overline{\cup\{S(e,n): n \leq m, q(S(e,m)) \neq q(S(e,n))\}}$. We note that

iv) $\hat{S}(e,m)$ is open,

v) $S(e,m) \cap Y = \hat{S}(e,m) \cap Y$,

vi) $\hat{S}(e,m) \cap \hat{S}(e',m') \neq \emptyset$ implies $q(S(e,m)) = q(s(e',m'))$.

The last statement follows from the definition of $\hat{S}$ because either $m \leq m'$ or $m' \leq m$.

Let $U = \cup\{S(e,m): e \in E, m < \omega, q(S(e,m)) = H\}$; let $V_i = \cup\{S(e,m): e \in E, m < \omega, q(S(e,m)) = K_i\}$. We claim that

$\{U\} \cup \{V_i: i \in I\}$ satisfies the conclusion of cwr. By iv) it is an open family; by vi) it is a disjoint family. That $U \supset H$ and $V_i \supset K_i$ for all i, follows from v), the fact that we always considered covers of X, and $\cup\mathcal{S}_i \cap Y = K_i$. This completes the proof of Theorem 1.6.

2. Spaces with σ-Locally Countable Bases

The first result about these spaces is Theorem 2.1 a), due to Fedorčuk. Aull [A] pointed out that the same proof converts covering properties to base properties, explicitly mentioning 2.1 a)-d), some of which were also noted by Shiraki [S]. To this list we add one more, 2.1 e), which can be strengthened to Corollary 2.2.

2.1. *Theorem. Suppose* X *has a* σ-*locally countable base.*

a) If X *is paracompact, then* X *is metrizable.*

b) If X *is* σ-*metacompact, then* X *has a* σ-*point finite base.*

c) If X *is screenable, then* X *has a* σ-*disjoint base.*

d) If X *is weakly* Θ *refinable, then* X *has a* Θ *base.*

e) If X *is subparacompact, then* X *is a* σ *space.*

Proof. We do only a) as the others are the same. Let $\mathcal{B} = \cup\{\mathcal{B}_n: n \in \omega\}$ be a base for X, where each $\mathcal{B}_n$ is locally countable. Let $\mathcal{W}_n$ be an open cover witnessing that $\mathcal{B}_n$ is locally countable, i.e. each $W \in \mathcal{W}_n$ meets only countably many elements of $\mathcal{B}_n$. By paracompactness let $\mathcal{R}_n$ be a locally finite refinement of $\mathcal{W}_n$. Because $\mathcal{W}_n$ witnesses that $\mathcal{B}_n$ is locally countable, for every $R \in \mathcal{R}_n$ we may write $\{B \in \mathcal{B}_n: R \cap B \neq \emptyset\}$ as $\{B(R,m): m \in \omega\}$. Because $\mathcal{R}_n$ is locally finite, $\mathcal{V}_{n,m} = \{B(R,m) \cap R: R \in \mathcal{R}_n\}$ is locally finite. Thus X has a σ-locally finite base.

2.2. *Corollary. A subparacompact space* X *with a σ-locally countable base is a Moore space.*

Proof. Immediate (with a wide knowledge of generalized metrizable spaces). A subparacompact space is weakly θ refinable so by 2.1 d) X has a θ base. By 2.1 e) X is a σ space, hence perfect. A perfect space with θ base is a Moore space.

Alternatively, a first countable σ space is semimetrizable, and a semimetrizable space with a point countable base is a Moore space.

The only other positive result we know about spaces with a σ-locally countable base is that they have a dense metrizable subset. This follows quickly from the following theorem of H. E. White [Wh] (what White calls a pseudobase, Juhász called a π base).

2.3. *Theorem. Let* X *be a first countable (Hausdorff) space.* X *has a dense metrizable subspace iff* X *has a σ-disjoint pseudobase.*

2.4. *Corollary. A space* X *with a σ-locally countable base has a dense metrizable subspace.*

Proof. Let $\mathcal{B} = \cup\{\mathcal{B}_n: n \in \omega\}$ be a base for X with $\mathcal{B}_n$ locally countable. Let $\mathcal{W}_n$ be a maximal family of disjoint open sets such that each $W \in \mathcal{W}_n$ meets only countably many elements of $\mathcal{B}_n$. So, for each $W \in \mathcal{W}_n$ we may write $\{B \in \mathcal{B}_n: B \cap W \neq \emptyset\}$ as $\{B(W,m): m \in \omega\}$. Set $\Psi_{m,n} = \{W \cap B(m,n): W \in \mathcal{W}_n\}$, a family of disjoint open sets. Thus $\cup\{\Psi_{m,n}: m,n \in \omega\}$

is a σ-disjoint pseudobase, and X has a dense metrizable subset.

We now describe an example of a nonmetrizable Moore space with a σ-locally countable base, and then a non-Moore space with a σ-locally countable space.

2.5. *Example. A Moore space* X *which*

i) has a σ*-locally countable base*

ii) is the union of two open metrizable subsets

iii) is not collectionwise Hausdorff.

Hence X *is none of metrizable, normal, paralindelöf, and countably paracompact.*

Proof of hence. normal + ii) $\to$ metrizable $\to$ collectionwise Hausdorff; countably paracompact + i) $\to$ paralindelöf $\to$ collectionwise Hausdorff.

The point set of X will be $A \cup B \cup C$, where $A = \{(\alpha): \alpha < \omega_1\}$, $B = \{(n,\beta): n < \omega,\ \beta < \omega_1\}$, $C = \{(\alpha,\beta,n): \alpha < \beta < \omega_1, n < \omega\}$. To visualize this example, think of A as a subset of the x-axis, and the set $\{(\alpha,\beta,n): \alpha < \beta < \omega_1\}$, for fixed α, as a subset of an interval of length $1/n$ on the line $y = 1/n$, centered at $(\alpha,1/n)$. These sets are disjoint as varies. For fixed n and β, the points $\{(\alpha,\beta,n): \alpha < \beta\}$ are essentially a sequence converging to $(n,\beta) \in B$.

Points of C are isolated. For $(n,\beta) \in B$, F a finite subset of $\{\alpha: \alpha < \beta\}$, $U(n,\beta,F) = \{(n,\beta)\} \cup \{(\alpha,\beta,n) \in C: \alpha \notin F\}$ is a basic open set. For $(\alpha) \in A$, $m < \omega$, $V(\alpha,m) = \{(\alpha)\} \cup \{(\alpha,\beta,n) \in C: \alpha < \beta < \omega_1, n \geq m\}$ is a basic open set. It is routine to verify that the basic open sets are clopen; hence X is regular. It is also easy to see that

$A \cup C$ and $B \cup C$ are disjoint unions of basic open sets, hence X is the union of two open metrizable subsets.

Since $(n,\beta) \neq (n',\beta')$ implies $U(n,\beta,F) \cap U(n',\beta',F') = \emptyset$, and $\alpha \neq \alpha'$ implies $V(\alpha,m) \cap V(\alpha',m') \neq \emptyset$ it is easy to see that X has a σ-disjoint base. Showing that X has a σ-locally countable base requires some bookkeeping. For $\beta < \omega_1$, enumerate the finite subsets of $\{\alpha: \alpha < \beta\}$ as $\{F(\beta,k): k < \omega\}$. For $\beta < \omega_1$ also enumerate $\{\alpha: \alpha < \beta\}$ as $\{\alpha(\beta,\ell): \ell < \omega\}$. Let

$$\mathcal{B}(\ell,n) = \{\{(\alpha(\beta,\ell),\beta,n)\}: \beta < \omega_1\}.$$

$$\mathcal{B}'(k,n) = \{U(n,\beta,F(\beta,k)): \beta < \omega_1\}.$$

$$\mathcal{B}''(m) = \{V(\alpha,m): \alpha < \omega_1\}.$$

Clearly, there are countably many $\mathcal{B}$'s and the union of all of them is a basis for X. Each $\mathcal{B}$ is disjoint collection of open sets, so with respect to the points in its union, $\mathcal{B}$ is locally one. Further, $\cup\mathcal{B}(\ell,n)$ and $\cup\mathcal{B}'(k,n')$ are closed, so with respect to the points of their complement, they are locally zero. $\cup\mathcal{B}''(m)$ is not closed, for if $n \geq m$, then $(n,\beta) \in \cup\mathcal{B}''(m)$. We notice that $V(\alpha,m) \cap U(n,\beta,\emptyset) \neq \emptyset$ iff $n \geq m$ and $\alpha < \beta$. So $U(n,\beta,\emptyset)$ meets only countably many elements of $\mathcal{B}''(m)$. Thus $\mathcal{B}''(m)$ is locally countable and X has a σ-locally countable base.

$A \cup B$ is a closed discrete subset of X. Aiming to show that X is not collectionwise Hausdorff, let $\{V(\alpha,m_\alpha): (\alpha) \in A\} \cup \{U(n,\beta,F_{\beta,n}): (n,\beta) \in B\}$ be a family of open sets. Now some $\overline{m} \in \omega$ must be m_α for uncountably many α's. Then for some $\overline{\beta} < \omega_1$ $\{\alpha < \beta: m_\alpha = \overline{m}\}$ is infinite, and thus there is $\overline{\alpha} \in \{\alpha < \beta: m_\alpha = \overline{m}\} - F_{\overline{\beta},\overline{m}}$. So $V(\overline{\alpha},\overline{m}) \cap U(\overline{m},\overline{\beta},F_{\overline{m},\overline{\beta}}) \neq \emptyset$, and X is not cwH.

The next example is due to Gary Gruenhage, [DGN].

2.6. *Example. A space Z which*

i) has a σ-locally countable base

ii) is not countable metacompact.

Hence Z is not perfect, θ refinable, nor Moore.

Let $\mathbf{c}$ be the initial ordinal of cardinality continuum. Let M be the set of all functions from ω to $\mathbf{c}$; let Σ be the set of all functions from a natural number to $\mathbf{c}$. For $\sigma \in \Sigma$, define $[\sigma] = \{m \in M: \sigma \subset M\}$. Then M has a natural completely metrizable topology with σ-discrete base--$\{[\sigma]: \sigma \in \Sigma\}$.

We will define a set F of functions f_α from ω to Σ. The point set of our space, Z, will be $M \cup F$. M will be an open set with its natural topology. F will be a closed discrete set. The basic open sets meeting F are $B(\alpha,n) = \{f_\alpha\} \cup \bigcup\{[f_\alpha(m)]: m \geq n\}$. Z thus resembles the pseudocompact not compact Moore space Ψ, or a tree space. There are two types of points--"good" and "bad." Each "bad" point p is associated with a sequence of basic open sets of "good" points; a neighborhood of p includes all but finitely many of these basic open sets. Our task is to define F small enough that Z is regular and has a σ-locally countable base, but large enough that Z is not countably metacompact.

Let $\{S_\alpha: \alpha < \mathbf{c}\}$ be a well ordering of the set of functions from ω to Σ satisfying, for all $n < m < \omega$,

1) $S_\alpha(n)(0) \neq S_\alpha(m)(0)$

2) $\operatorname{dom} S_\alpha(n) = \operatorname{dom} S_\alpha(m) = k_\alpha$.

Now we define by induction on $\alpha < \mathbf{c}$, functions from ω to Σ satisfying, for all $\alpha < \beta < \mathbf{c}$, for all $n < m$, $p < \omega$,

3) $S_\alpha(n) \subset f_\alpha(n)$

4) dom $f_\alpha(n) = k_\alpha + n + 1$

5) if dom $f_\alpha(p) > k_\beta$, then $[f_\alpha(p)] \cap [f_\beta(m)] = \phi$

6) if $k_\alpha = k_\beta$, then $[f_\alpha(p)] \cap [f_\beta(m)] = \phi$.

This construction can be carried out because at each stage there are $\mathfrak{c}$ possibilities of which only less than $\mathfrak{c}$ have been forbidden. Let $\mathcal{B}_n = \{[\sigma]: \text{dom } \sigma = n\}$, let $\mathcal{B}_{n,k} = \{B(\alpha,n): k_\alpha = k\}$. It is straightforward to verify by cases that each $\mathcal{B}$ is a disjoint, locally countable family of clopen sets. Thus Z is a regular space with a σ-disjoint, σ-locally countable base.

Aiming towards showing that Z is not countably metacompact, let us define $F_n = \{f_\alpha: k_\alpha \geq n\}$. Then F_n is a decreasing sequence of closed sets with $\cap\{F_n: n < \omega\} = \phi$. Suppose U_n is open, $U_n \supset F_n$. For $n \leq k < \omega$, let $A_{n,k} = \{\sigma \in \Sigma:$ dom $\sigma = k$, $[\sigma] \cap U_n = \phi\}$. We claim that $W_{n,k} = \{\sigma(0): \sigma \in A_{n,k}\}$ is finite. For if $W_{n,k}$ were infinite there would be an S_α with ran $S_\alpha \subset A_{n,k}$, and U_n would not contain a neighborhood of $f_\alpha \in F_n$. Let $W = \cup\{W_{n,k}, n \leq k < \omega\}$, and let $M' = \{m \in M: m(0) \notin W\}$. M' is a nonempty (because W is countable) completely metrizable space, and for each n, $U_n \cap M'$ is dense open in M'. Then by Baire's Theorem, $\cup\{U_n: n \in \omega\} \neq \phi$, and Z is not countably metacompact.

2.7. *Remarks*. Because X has a σ-disjoint base, X is screenable and metacompact. The argument showing that X is not collectionwise Hausdorff also shows that X is not pseudonormal. The product of countably many copies of X has the same properties as X except every nonempty open set is nonmetrizable.*

The argument that Z is not countably metacompact also shows that Z is not collectionwise Hausdorff. For if we could screen F, we could define $U_n \supset F_n$, $\cap\{U_n: n \in \omega\} = \phi$.

Spaces with the same properties as X and Z except that they are collectionwise Hausdorff can be constructed assuming the existence of an E set (see [F], [P] for analogous constructions).

In some sense, Z is the simplest possible example of a nonperfect space S with a σ-disjoint, σ-locally countable base. For by Theorem 4.1 card S $\geq$ **c** (if S is to be an honest example). The following lemma shows that the complement of the "bad" closed set cannot be discrete.

2.8. *Lemma. In a space* X *with a σ-locally countable base, the set* I *of isolated points is* F_σ.

Proof. Let $I_n = \{x: \{x\} \in \mathcal{B}_n\}$ where each $\mathcal{B}_n$ is locally countable and $\cup\{\mathcal{B}_n: n \in \omega\}$ is a base for X. Let $\mathcal{W}_n$ be an open cover of X satisfying

1) for each $W \in \mathcal{W}_n$, $\{x \in I_n: x \in \mathcal{W}\}$ is countable

2) for each $x \in I_n$, $\{W \in \mathcal{W}_n: x \in W\}$ is countable.

Now use the method of countable equivalence classes, as in Theorem 1.7.

3. Implications Between Paralindelöf and σ-Locally Countable Bases

As we noted in the introduction, the (nontrivial) implications in metrizable ↔ σ-locally finite base

*By other results, X^ω is not the union of countably many closed metrizable subspaces.

$\to$ σ-paracompact $\leftrightarrow$ paracompact are among the most beautiful and most useful theorems in general topology. While there are many nice spaces that are paracompact but not metrizable, paracompact + Moore, or more generally, paracompact + base of countable order, implies metrizable. In this section we examine these implications with "finite" replaced by "countable" and "paracompact" replaced by "paralindelöf."

Of course the trivial implications paralindelöf $\to$ σ-paralindelöf and σ-locally countable base $\to$ σ-paralindelöf hold. Moreover, any paracompact nonmetrizable space is a paralindelöf space without a σ-locally countable base. It is easy to see that paralindelöf + Moore $\to$ σ-locally countable base, but with "base of countable order" replacing "Moore" it is unknown. Antiparallel to the implication σ-paracompact $\to$ paracompact, there are σ-paralindelöf not paralindelöf spaces (e.g. Examples 2.5 and 2.6). The main result of this section, due to Frank Tall [T], is that the addition of countably paracompactness restores the parallel.

3.1. *Theorem. A countably paracompact, σ-paralindelöf space* X *is paralindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X. Let $\mathcal{V}_n$ be locally countable with $\cup\{\mathcal{V}_n: n \in \omega\}$ an open refinement of $\mathcal{U}$. Now $\{\cup\mathcal{V}_n: n \in \omega\}$ is a countable open cover of X, so by countable paracompactness, there is a locally finite open cover $\mathcal{W} = \{W_n: n \in \omega\}$ of X with $W_n \subset \cup\mathcal{V}_n$.

We claim $\mathcal{R} = \{W_n \cap V: V \in \mathcal{V}_n, n \in \omega\}$ is a locally countable open refinement of $\mathcal{U}$. Clearly it is open, and it refines $\mathcal{U}$ by our definition of $\mathcal{V}_n$. Let $x \in X$, we must show that x has a neighborhood N with $\{R \in \mathcal{R}: R \cap N \neq \phi\}$ countable. First, since $\mathcal{W}$ is locally finite, x has a neighborhood N' such that $W_n \cap N' \neq \phi$ implies $n \in F$, a finite set. Because $\mathcal{V}_n$ is locally countable, x has a neighborhood N_n meeting only countably many elements of $\mathcal{V}_n$. Set $N = N' \cap \cap \{N_n: n \in F\}$.

4. Spaces of Cardinality ω_1

Let us contrast Examples 2.5 and 2.6. Both are non-metrizable spaces with a σ-locally countable base. Example 2.5 is metacompact and perfect and has cardinality ω_1, Example 2.6 is not even countably metacompact and has cardinality $\mathfrak{c}$. We ask whether these differences are related. Is it provable in ZFC that there is a space of cardinality ω_1 with a σ-locally countable base which is not perfect or not metacompact? We may ask the same question with "σ-locally countable" replaced by "σ-point finite" or "point countable." In this section we show the answer is no in the "σ-point finite" case. Peter Davies has shown in ZFC that there is a non-perfect regular space of cardinality ω_1 with a point countable base.

The following theorem is a strengthening of the theorem that MA + ¬CH implies that there is a Q set. Note that example 2.2 shows that we cannot include normal in the conclusion.

4.1. *Theorem.* (MA) *If* X *is a* T_1 *space of cardinality less than* $\mathfrak{c}$ *with a σ-point finite base, then every subset of* X *is a* G_δ *subset of* X. *Further,* X *is metacompact.*

Proof. Let Y be an arbitrary subset of X. Let $\mathcal{B} = \cup\{\mathcal{B}_n: n \in \omega\}$ be a base for X such that for all $x \in X$, all $n \in \omega$, $\{B \in \mathcal{B}_n : x \in B\}$ is finite. Define

$\mathcal{P} = \{(\mathcal{S},a) : \mathcal{S}$ is a finite sequence $(S_0,\cdots,S_k)$ where each S_m is a finite union of elements of $\mathcal{B}$ and a is a finite subset of $X - Y\}$.

$(\mathcal{S},a) \leq (\mathcal{S}',a')$ iff for each m $S_m \supset S'_m$, $a \subset a'$, and $a' \cap \cup\{S_m - S'_m : m < \omega\} = \emptyset$.

Because X is T_1, for each $z \in X - Y$, $D_z = \{(\mathcal{S},a) \in \mathcal{P}: z \in a\}$ is dense. Similarly, for each $m \in \omega$, $y \in Y$, $D_{m,y} = \{(\mathcal{S},a) \in \mathcal{P}: y \in S_m\}$ is dense. Thus, after we have established that $\mathcal{P}$ is ccc, then by MA we may conclude that there is a filter $G \subset \mathcal{P}$ meeting each of these $<\mathfrak{c}$ dense sets. Then we define $U_m = \cup\{S_m : S_m \in \mathcal{S}, (\mathcal{S},a) \in G\}$. Clearly U_m is open. and since G meets each of the above dense sets, for all m, $U_m \supset Y$ and $\cap\{U_m : m \in \omega\} \subset Y$.

Thus, the proof boils down to demonstrating that $\mathcal{P}$ is ccc. Aiming for a contradiction, we assume that $W \subset \mathcal{P}$, cardinality of $W = \omega_1$, and elements of W are pairwise incompatible. By successively applying the pigeonhole principle and the Δ system lemma, we can find $\{(\mathcal{S}^\alpha,a^\alpha) : \alpha < \omega_1\} \subset W$ such that for all α

i) $\mathcal{S}^\alpha = (S_0^\alpha,\cdots,S_k^\alpha)$

ii) for all $m \leq k$, $S_m^\alpha = B(\alpha,m,0) \cup \cdots \cup B(\alpha,m,j(m))$

iii) for all $m \leq k$, $i < j(m)$, $B(\alpha,m,i) \in \mathcal{B}_{n(m,i)}$

iv) $a^\alpha = \{z(\alpha,p) : p \leq \ell\}$

v) $\{a^\alpha : \alpha < \omega_1\}$ is a Δ system with root $\{z(\alpha,p) : p \leq \ell'\}$

vi) for all $n < \omega$, $p \leq \ell$ card$\{B \in \mathcal{B}_n : z(\alpha,p) \in B\} = r(n,p)$.

We notice that if $\alpha < \beta$, $(\mathcal{S}^\alpha,a^\alpha)$ and $(\mathcal{S}^\beta,a^\beta)$ are incompatible then for some p, m, i, e, satisfying $\ell' < p \leq \ell$, $m \leq k$, $i \leq j(m)$, $e \in \{0,1\}$ we have condition (e,p,m,i) where (o,p,m,i) means "$z(\alpha,p) \in B(\beta,m,i) - B(\alpha,m,i)$"

(1,p,m,i) means "$z(\beta,p) \in B(\alpha,m,i) - B(\beta,m,i)$".

Now we apply the Erdös-Rado-Milnor theorem $\omega_1 \to (\omega_1,\omega)_\omega$, where the partition is indexed by {compatible} $\cup\{(e,p,m,i): e < 2, \ell' < p \leq \ell, m \leq k, i \leq j(m)\}$ to get either ω_1 compatible elements, or ω elements satisfying some condition (e,p,m,i). In the latter case we violate vi), so the former case holds. But this contradicts our assumption that the elements of W are pairwise incompatible, establishing the first claim of the theorem.

The proof that X is metacompact is parallel to the above proof, and so is omitted. The proofs of the following Corollaries follow the proof of Corollary 2.2.

4.2. *Corollary.* (MA) *A space of cardinality less than* **c** *with a σ-point finite base and a σ-locally countable base is a Moore space.*

4.3. *Corollary.* (MA) *A metacompact space of cardinality less than* **c** *with a σ-locally countable base is a Moore space.*

5. Questions

Recall our convention that all spaces are T_1 and regular.

1. Is a paralindelöf space paracompact?

2. Assuming the answer to 1 is no, what about interpolants? For example, does paralindelöf imply metacompact? Does paralindelöf plus metacompact imply paracompact? Similarly for normal, collectionwise normal, countably paracompact, etc.

3. Is a sL-cwH, metacompact space paralindelöf?

4. Is a collectionwise normal space with a σ-locally countable base metrizable?

5. Does a paralindelöf space with a base of countable order have a σ-locally countable base? (Yes, if yes to 1.)

6. Is there an honest (i.e. provable from ZFC) example of a space of cardinality ω_1 with a σ-locally countable base which is not perfect? not metacompact?

7. If X has a σ-disjoint base $\mathcal{B}$, and a σ-locally countable base $\mathcal{B}'$ must X have a base $\mathcal{B}''$ which is simultaneously σ-disjoint and σ-locally countable?

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