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1. Introduction

In the study of fixed point theorems for a continuous map $f: X \rightarrow X$, X a compact connected metric ANR, there are several interesting numbers associated with the map f . Namely, they are the Lefschetz number $L(f)$, the Nielsen number $N(f)$, and the fixed point index. In many cases, the Nielsen number gives more complete geometric information than the Lefschetz number. For example, if f is a map on an n -manifold of dimension ≥ 3 such that $N(f) = 0$, then there is a fixed point free map g that is homotopic to f , [3]. This is not implied by the vanishing of the Lefschetz number. The difficulty with the Nielsen number is that it is often hard to compute. In an effort to devise a new tool to compute the Nielsen number of a continuous map, Brown ([4], [5]) initiated the study of fiber-preserving maps.

Let $J = \{E, p, B\}$ be a Hurewicz fibering with a regular lifting function λ and with E , B and all fibers compact, connected, metric ANR's. If $f: E \rightarrow E$ is a fiber-preserving map, then it induces a unique map $\bar{f}: B \rightarrow B$ such that $pf = \bar{f}p$ and a map $f_b: p^{-1}(b) \rightarrow p^{-1}(b)$ for each $b \in B$, defined by $f_b(e) = \lambda(f(e), \omega)(1)$ for each $e \in p^{-1}(b)$ and a path ω from $\bar{f}(b)$ to b in B . It is known that if J is orientable, then the Euler-Poincaré characteristics of the spaces involved satisfy $\chi(E) = \chi(B)\chi(p^{-1}(b))$, ([16]), p. 481). In [4] and

[11], a similar result is proved for the Lefschetz numbers of maps; i.e., $L(f) = L(\bar{f}) \cdot L(f_b)$, $b \in B$. Note that this equality reduces to the previous one if f is the identity map on E . Unfortunately, a similar product theorem does not hold for the Nielsen numbers and for the fixed point indices of maps, as was discovered by Brown and Fadell in [6]. However, they were able to exhibit various conditions on $\mathcal{J} = \{E, p, B\}$ under which the relation $N(f) = N(\bar{f}) \cdot N(f_b)$, $b \in B$, did hold. It is shown that there is an integer $\mathcal{P}(\mathcal{J}, f)$ such that $N(f) \mathcal{P}(\mathcal{J}, f) = N(\bar{f}) \cdot N(f_b)$ for a locally trivial fibering $\mathcal{J} = \{E, p, B\}$, where E , B and all fibers satisfy the so-called "Jiang condition" [14]. The integer $\mathcal{P}(\mathcal{J}, f)$ is determined by the homomorphisms $f_{\#}: \pi_1(E) \rightarrow \pi_1(E)$, $f_{b\#}: \pi_1(p^{-1}(b)) \rightarrow \pi_1(p^{-1}(b))$ induced by f and f_b , respectively. Interpreting $\mathcal{P}(\mathcal{J}, f)$ as an "obstruction" to a product theorem (i.e., $N(f) = N(\bar{f}) \cdot N(f_b)$ if and only if $\mathcal{P}(\mathcal{J}, f) = 1$) several additions to the collection of fiber spaces for which a product theorem holds were given. Some generalizations of this result are given also in [9] and [12].

In this paper, our primary purpose is to give a general analysis of the relation among the Nielsen numbers of a "fiber map triple" (f, \bar{f}, f_b) without any conditions on the spaces involved. This analysis is summarized in Theorem (2.1). Applying this, we prove a general product theorem for Nielsen numbers which clarifies and extends many of the previously known facts. We also consider fiber spaces where all the spaces satisfy the Jiang condition and give an analogous theorem for the fixed point indices together with

some concrete applications. As a by-product of the proof of this theorem, we are able to give a geometric interpretation to the obstruction $\mathcal{P}(\mathcal{J}, f)$, which may be useful for the computation of Nielsen numbers.

Finally, the authors would like to thank E. Fadell for his helpful communications and preprints.

2. The Main Theorem

Let $\mathcal{J} = \{E, p, B\}$ be an orientable Hurewicz fibering with a regular lifting function λ . A Hurewicz fibering is orientable if, for every loop ω based at $b \in B$, the induced map $\bar{\omega}: p^{-1}(b) \rightarrow p^{-1}(b)$, defined by $\bar{\omega}(e) = \lambda(e, \omega)(1)$, $e \in p^{-1}(b)$, is homotopic to the identity map on $p^{-1}(b)$. We assume that all spaces, including fibers, are compact, connected, metric, ANR's and consider fiber preserving maps $f: E \rightarrow E$ which have non-vanishing Lefschetz numbers. We denote the maps induced by f on the base space B and on a fiber $p^{-1}(b)$ by \bar{f} and f_b , respectively. For a map $g: X \rightarrow X$, $N(g)$ will denote the Nielsen number of g and $\Phi(g)$ will denote the fixed point set of g . It is a classical fact [3] that homotopic maps have the same Nielsen numbers. Fadell [8] showed that, for an orientable Hurewicz fibering, the Nielsen number $N(f_b)$ of the map $f_b: p^{-1}(b) \rightarrow p^{-1}(b)$, $b \in B$, is well defined; i.e., it is independent of the choice of path ω from $\bar{f}(b)$ to b and the point $b \in B$. Hence we may define the Nielsen number of f on fiber, $N_F(f)$, to be $N(f_b)$ for some $b \in B$.

We denote the essential fixed point classes of f [3] by F_E^1, \dots, F_E^m , $m = N(f)$, and the essential fixed point classes of \bar{f} by F_B^1, \dots, F_B^r , $r = N(\bar{f})$. For each $b \in \Phi(\bar{f})$, let

$K_b^i = F_E^i \cap p^{-1}(b)$, $i = 1, \dots, m$, and let $\#K_b^i$ denote the number of essential fixed point classes for f_b contained in K_b^i . We note that if $x \in K_b^i$, $y \in K_b^j$, $i \neq j$, then x and y are not equivalent for f_b . Two points $x, y \in \Phi(f_b)$ are equivalent if there is a path C from x to y in $p^{-1}(b)$ that is homotopic to $f_b(C)$ with endpoints fixed. Since $N(f_b)$ is independent of the choice of path [8], it may be assumed without loss of generality that f_b is the restriction of f to $p^{-1}(b)$ for each $b \in \Phi(\bar{F})$.

A space X is called a Jiang space or is said to satisfy the Jiang condition if the Jiang subgroup $T(X)$ (see [3]) of X is equal to the fundamental group of X ; i.e., $T(X) = \pi_1(X)$. It is well known ([3], [10]) that if a compact, connected, ANR space X satisfies the Jiang condition, then $\pi_1(X, x_0)$ is abelian and all the fixed point classes of any map $f: X \rightarrow X$ have the same indices. If $i(f)$ denotes this common number, then $L(f) = i(f) \cdot N(f)$ for a map f on a Jiang space.

We now state the main theorem.

Theorem (2.1). Given an orientable Hurewicz fibering $J = \{E, p, B\}$ with E, B and all fibers compact, connected, metric, ANR's, and given a fiber-preserving map $f: E \rightarrow E$ such that $L(f) \neq 0$,

(a) *the Nielsen number $N(f) = \lambda_1 + \dots + \lambda_r$, where $r = N(\bar{F})$ and λ_j is the number of essential fixed point classes of f having non-empty intersection with $p^{-1}(F_B^j)$, $j = 1, \dots, r$,*

(b) *for each $b \in F_B^j$, $N(f_b) \geq \#K_b^1 + \dots + \#K_b^r$ with equality holding if either all fibers are Jiang spaces or*

$F_E \cap p^{-1}(b)$ is at most one fixed point class for f_b for each fixed point class F_E of f . (Here the essential fixed point classes of f are numbered so that F_E^1, \dots, F_E^j are exactly those meeting $p^{-1}(b)$.)

The key step in the proof of the theorem is the following lemma.

Lemma (2.2). Let F_E be a fixed point class of f and let F_B be the fixed point class of \bar{f} containing $p(F_E)$. For each $b \in F_B$, let F_{Y_1}, \dots, F_{Y_k} be the fixed point classes for $f_b = f|_{p^{-1}(b)}$ contained in $F_E \cap p^{-1}(b)$. Then

$$1) i(F_E) = i(F_B) \cdot \sum_{n=1}^k i(F_{Y_n}).$$

2) If $p(F_E) \subsetneq F_B$, then $i(F_{Y_n}) = 0$, $1 \leq n \leq k$, and so $i(F_E) = 0$.

Thus if F_E is essential, then $p(F_E) = F_B$.

This lemma is an obvious extension of the argument given by Brown in [5, p. 492]. His argument shows that the result can be stated as above in a general situation. The extension of Brown's argument to a Hurewicz fibering with spaces compact polyhedra is also obvious by the following recent result of Fadell [8].

Proposition (Fadell [8]). If

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow h & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

is a homotopy commutative diagram of maps on compact, ANR spaces X and Y , where h is a homotopy equivalence, then

$N(f) = N(g)$. (A special case of this result is contained in Jiang's paper [10].)

Furthermore, to handle the case when spaces are compact, metric, ANR's, we may make use of the result of R. Miller [13] and J. West [17] that a compact, metric, ANR space is the same homotopy type as a finite polyhedron.

We also note that Lemma (2.2) applied to a locally trivial fiber space with all spaces compact polyhedra together with a result of Fadell [9] can be used to prove Theorem (2.1) for a Hurewicz fiber space with all spaces compact, metric, ANR's. Fadell used the closed fiber smoothing theorem of A. Casson and D. Gottlieb to establish the following theorem.

Theorem (Fadell [9]). Let $\mathcal{J} = \{E, p, B\}$ be a Hurewicz fibering with E and B compact, metric, ANR's and $f: E \rightarrow E$ a fiber-preserving map. Then there is a locally trivial fibering $\mathcal{J}' = \{E', p', B'\}$ with fiber F' and a fiber-preserving map $g: E' \rightarrow E'$ with the following properties:

- 1) B' and F' are compact polyhedra.
- 2) $\bar{g}: B' \rightarrow B'$ has precisely $N(\bar{g})$ fixed points, each in a maximal simplex.
- 3) For each $b \in \Phi(\bar{g})$, $g_b: F' \rightarrow F'$ has precisely $N(g_b)$ fixed points, each in a maximal simplex of F' .
- 4) $N(f) = N(g)$, $N(\bar{f}) = N(\bar{g})$, $N_{F'}(f) = N_{F'}(g)$.

Proof of Theorem (2.1). Let F_B^j be any essential fixed point class of \bar{f} and let ℓ_j be the number of essential classes of f meeting $p^{-1}(F_B^j)$. We note that if F_E^k is one of

these classes, it follows from the fact that $p(F_E^k) = F_B^j$ that $F_E^k \cap p^{-1}(b) \neq \emptyset$ for all $b \in F_B^j$, and so ℓ_j is exactly the number of essential classes of f intersecting $p^{-1}(b)$ for all $b \in F_B^j$. Now it follows from Lemma (4.1) of [5] and [8] that if $p(e)$ is not equivalent to $p(e') \in \Phi(\bar{f})$, then e and e' are not equivalent, $e, e' \in \Phi(f)$. Furthermore, one sees from (1) of Lemma (2.2) that the essentiality of F_E implies the essentiality of $F_B = p(F_E)$ and so each essential class of f lies in $p^{-1}(F_B^j)$ for some j , $1 \leq j \leq r = N(\bar{f})$. Therefore, $N(f) = \ell_1 + \dots + \ell_r$. To verify part (b) of the theorem, let $b \in F_B^j$ and let $F_E^1, \dots, F_E^{\ell_j}$ be the essential classes of f having non-empty intersection with $p^{-1}(F_B^j)$. Since $f_b \equiv f|_{p^{-1}(b)}$, if e and e' are equivalent in $\Phi(f_b)$, then they are equivalent in $\Phi(f)$; that is, if $e \in K_b^i = F_E^i \cap p^{-1}(b)$ and $e' \in K_b^j = F_E^j \cap p^{-1}(b)$ and $i \neq j$, then e and e' are not equivalent for f_b . Thus f_b has at least $\#K_b^1 + \dots + \#K_b^{\ell_j}$ essential fixed point classes. To prove the equality we must show that if F_E is an unessential class of f meeting $p^{-1}(b)$, then F_E contributes only unessential classes to $\Phi(f_b)$.

Case (1). By (1) of Lemma (2.2), $i(F_E) = i(F_B^j) \cdot \sum_{n=1}^k i(F_{Y_n})$. Since by assumption $i(F_E) = 0$ while $i(F_B^j) \neq 0$, one has $\sum_{n=1}^k i(F_{Y_n}) = 0$. Moreover, if $p^{-1}(b)$ is a Jiang space, then we have $\sum_{n=1}^k i(F_{Y_n}) = k \cdot i(F_{p^{-1}(b)})$, where $i(F_{p^{-1}(b)})$ is the common index for the fixed point classes of f_b . Hence $0 = i(F_{p^{-1}(b)}) = i(F_{Y_n})$, $1 \leq n \leq k$; i.e., each F_{Y_n} is an unessential class.

Case (2). As in Case (1), one sees that $0 = \sum_{n=1}^k i(F_{Y_n})$. By assumption, however, $k = 1$ and so $0 = i(F_{Y_1})$, as claimed.

This completes the proof of (2.1).

3. Product Theorems

In the first part of this section, we establish a new product theorem for the Nielsen numbers of a "fiber map triple" (f, \bar{f}, f_b) for a Hurewicz fibering $\mathcal{J} = \{E, p, B\}$. Using this result, examples of the calculation of certain Nielsen numbers will be given that were not previously obtainable.

Let $\mathcal{J} = \{E, p, B\}$ be an orientable Hurewicz fibering with all fibers, E and B compact, connected, metric, ANR's, and let $f: E \rightarrow E$ be a fiber-preserving map. With notation exactly as in Theorem (2.1) we prove the following theorem:

Theorem (3.1). Suppose, for each $b \in \Phi(\bar{f})$ and for each fixed point class F_E^i of f , $i = 1, \dots, m = N(f)$, that $K_b^i = F_E^i \cap p^{-1}(b)$ is either empty or at most one fixed point class of f_b . Then $N(f) = N(\bar{f}) \cdot N(f_b)$ for each $b \in B$. Conversely, if $N(f) = N(\bar{f}) \cdot N(f_b)$ for each $b \in B$, then $\#K_b^i \leq 1$ for each $b \in \Phi(\bar{f})$, $1 \leq i \leq N(f)$.

Proof. The first part follows immediately from Theorem (2.1). Indeed, if $\#K_b^i \leq 1$ for $1 \leq i \leq m$, for $b \in F_B^j$, one has by Lemma (2.2) and part (b) of Theorem (2.1) that $N(f_b) = \#K_b^1 + \dots + \#K_b^{\ell_j} = \ell_j$. Since $N(f_b)$ is independent of b [8], it follows that $\ell_1 = \ell_2 = \dots = \ell_r = N(f_b)$. Thus for $b \in \Phi(\bar{f})$, $N(f) = N(f_b) + \dots + N(f_b) = N(f_b) \cdot N(\bar{f})$.

To prove the converse, we write $N(f) = \underbrace{N(f_b) + \dots + N(f_b)}_r$, $r = N(\bar{f})$. Then for $b \in F_B^j$, Theorem (2.1) (b) says that $N(f_b) \geq \#K_b^1 + \dots + \#K_b^{\ell_j}$ when $p^{-1}(b) \cap F_E^i \neq \emptyset$, $1 \leq i \leq \ell_j$. Since (2.2) (1) implies that $\#K_b^i \geq 1$, this says that $N(f_b) \geq \ell_j$ for $b \in F_B^j$, $1 \leq j \leq r$. To prove the result, it will suffice to show that $N(f_b) = \ell_j$, $1 \leq j \leq r$. If, for

some k , $N(f_b) > \ell_k$, $b \in F_B^k$, then $N(f) = \underbrace{N(f_b) + \dots + N(f_b)}_r > \ell_1 + \dots + \ell_r$. This contradicts part (a) of Theorem (2.1).

Remark. Recently Fadell [9] has discovered a general condition on \mathcal{F} (\mathcal{F} admits a natural fiber splitting) under which it is possible to deform f so that each of its essential fixed point classes consists of one point and $N(f) = N(\bar{f}) \cdot N(f_b)$. While this result is certainly closely related to Theorem (3.1), the methods of proof of the two theorems are quite different and it does not appear that either result is a consequence of the other.

Let $\mathcal{F} = \{E, p, B\}$, $f: E \rightarrow E$ be as above. If $i_{\#}: \pi_1(p^{-1}(b), *) \rightarrow \pi_1(E, *)$ is injective (i the inclusion) and $\bar{f}_{\#}: \pi_1(B, *) \rightarrow \pi_1(B, *)$ fixes only the identity, then it is not hard to show that the hypothesis of Theorem (3.1) is satisfied ([9], (6.1)]. Hence we have $N(f) = N(f_b) \cdot N(\bar{f})$ for all $b \in B$. As a consequence one has the following example (compare (6.1) of [8] and Theorem 7 of [14]):

Example (3.2). Let $\mathcal{F} = \{E, p, P^m(\mathbb{R}), Y\}$ be a fiber bundle over real projective m -space $P^m(\mathbb{R})$, $m \geq 3$, with $Y = p^{-1}(b)$ a compact, connected, ANR. Let $f: E \rightarrow E$ be a fiber-preserving map. If the induced map \bar{f} on the base space is not homotopic to the identity map, then $N(f) = N(\bar{f}) \cdot N(f_b)$ for all $b \in B$. Furthermore, if m is odd, then $P^m(\mathbb{R})$ is a Jiang space and $N(\bar{f}) = 1$ and so $N(f) = N(f_b) = N(f|_{p^{-1}(b)})$. (Compare with (4.7) and (4.1) of [10].) In particular, if the fiber is S^1 and $f|_{S^1}$ has degree d , then $N(f_b) = |1-d| = N(f)$.

From now on we assume that all spaces involved in a fibering $\mathcal{J} = \{E, p, B\}$ are Jiang spaces and that any fiber-preserving map $f: E \rightarrow E$ has non-zero Lefschetz number; i.e., $L(f) \neq 0$. We consider to what extent the index of f , $i(f)$, may be computed from $i(\bar{f})$ and $i(f_b)$. To this end, we prove the following product theorem for the fixed point indices on Jiang spaces.

Theorem (3.3). *Let $\mathcal{J} = \{E, p, B\}$ be an orientable Hurewicz fibering with E , B and all fibers compact, connected, ANR's satisfying the Jiang condition, and let $f: E \rightarrow E$ be a fiber-preserving map. If an essential fixed point class of f meets each fiber $p^{-1}(b)$ in at most one fixed point class of f_b , then $i(f) = i(\bar{f}) \cdot i(f_b)$ for all $b \in B$.*

Proof. Let F_E be a fixed point class of f , F_B the fixed point class of \bar{f} containing $p(F_E)$, and let F_{Y_1}, \dots, F_{Y_k} be the fixed point classes for f_{b_0} in $F_E \cap p^{-1}(b_0)$, $b_0 \in F_B$. Then it is immediate, from lemma (2.2) and the fact that $p^{-1}(b_0)$ is a Jiang space, that $i(F_E) = i(F_B) \cdot i(F_Y) \cdot k$, where $i(F_Y)$ denotes the number $i(F_{Y_n})$, $n = 1, \dots, k$. Furthermore, since all spaces satisfy the Jiang condition, every fixed point class of f , \bar{f} and f_{b_0} has the same index $i(f) = i(F_E)$, $i(\bar{f}) = i(F_B)$ and $i(f_{b_0}) = i(F_Y)$ respectively for every such collection $(F_E; F_B; F_{Y_1}, \dots, F_{Y_k})$. Thus one has (with notation as in Theorem (2.1)) that $k = \#K_{b_0}^j$ for all $j = 1, \dots, m = N(f)$. Also, since both $N(f_b)$ and $L(f_b) = i(f_b) N(f_b)$ are independent of $b \in B$ ([4], [5] and [8]), it follows that $i(f_b)$ is independent of $b \in B$. By the relation $i(F_E) = i(F_B) \cdot i(F_Y) \cdot k$ one has that $k = \#K_b^j$, $j = 1, \dots, m$, for all $b \in \Phi(\bar{f})$, and hence

$k = \#K_b^i = i(f)/i(F) \cdot i(f_b)$. Thus if $\#K_b^j \leq 1$ for some (and hence every) F_E , it is immediate that $i(f) = i(\bar{f}) \cdot i(f_b)$. This completes the proof.

In (3.1), we showed that if $N(f) = N(\bar{f}) N(f_b)$ holds for each $b \in B$, then $\#K_b^i \leq 1$ for each $b \in \Phi(\bar{f})$, $1 \leq i \leq N(f)$. Therefore if spaces in a Hurewicz fibering $\mathcal{J} = \{E, p, B\}$ are Jiang spaces, then we get $i(f) = i(\bar{f}) \cdot i(f_b)$ for each $b \in B$. Hence applying [6] and (5.2) of [9], we get the following:

Corollary (3.4). Let $\mathcal{J} = \{E, p, B\}$ be an orientable Hurewicz fibering with E, B and all fibers compact, connected, ANR's satisfying the Jiang condition, and let $f: E \rightarrow E$ be a fiber-preserving map. Then $i(f) = i(\bar{f}) \cdot i(f_b)$ for each $b \in B$ if any one of the following holds:

1. \mathcal{J} admits a section $\sigma: B \rightarrow E$ such that $f\sigma = \sigma\bar{f}$ and $\pi_1(E)$ is abelian.
2. \mathcal{J} is trivial and $f = \bar{f} \times f_b$.
3. There is homotopy commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ f \downarrow & & \downarrow g \\ E & \xrightarrow{\Phi} & F \end{array},$$

where $F = p^{-1}(b)$, and $\Phi|_{p^{-1}(b)}$ is a homotopy equivalence for all $b \in B$.

Remark. According to Theorem 4 of [11], the number $\mathcal{P}(\mathcal{J}, f)$ is equal to the ratio $i(f)/i(\bar{f}) \cdot i(f_b)$, $b \in B$, so it follows from the proof of Theorem (3.3) that the invariant $\mathcal{P}(\mathcal{J}, f)$ is exactly the number of essential fixed point classes for f_b contained in a single essential fixed point class F_E with $F_E \cap p^{-1}(b) \neq \emptyset$. This is a geometric interpretation of

the number $\mathcal{P}(\mathcal{J}, f)$ which is defined in a purely algebraic manner for a locally trivial fibering $\mathcal{J} = \{E, p, B\}$ and a fiber-preserving map $f: E \rightarrow E$ in [14]. We also remark that since $N(f_b) = \lambda_j \cdot k$ for $b \in F_B^j$ the invariance of $N(f_b)$ over $b \in \Phi(f)$ implies that $\lambda_1 = \lambda_2 = \dots = \lambda_r = \lambda$. Hence Theorem (2.1) implies that $N(f) = \lambda \cdot N(\bar{f})$. In particular, $N(\bar{f})$ divides $N(f)$. This is a fact that does not follow from the relation $N(f) \mathcal{P}(\mathcal{J}, f) = N(\bar{f}) \cdot N(f_b)$.

Furthermore, the fact that $N(f_b) = \lambda \cdot k = \lambda \cdot \mathcal{P}(\mathcal{J}, f)$ and $N(f) = \lambda \cdot N(\bar{f})$ shows that Theorem 4 in [14] holds for a Hurewicz fibering.

Corollary (3.5) [15]. Let $\mathcal{J} = \{E, p, B\}$ be an orientable Hurewicz fibering with E, B and all fibers compact, connected, metric, ANR's satisfying the Jiang condition. If $f: E \rightarrow E$ is a fiber-preserving map with $L(f) \neq 0$, then $N(f) \cdot \mathcal{P}(\mathcal{J}, f) = N(\bar{f}) \cdot N(f_b)$ for all $b \in B$.

Corollary (3.6). Let $\mathcal{J} = \{E, p, P^n(\mathbb{C})\}$ be a principal T^k -bundle over an n -dimensional, complex projective space $P^n(\mathbb{C})$ and $f: E \rightarrow E$ be a fiber-preserving map. If for some j with $1 \leq j \leq N(f)$, $F_E^j \cap p^{-1}(b)$ is either empty or a single fixed point class of f_b for each $b \in \Phi(\bar{f})$, then $i(f) = \pm i(\bar{f})$.

Proof. According to Corollary 12 of [14], each total space of a principal T^k -bundle over $P^n(\mathbb{C})$ satisfies the Jiang condition. Thus $i(f) = i(\bar{f}) \cdot i(f_b) \cdot \mathcal{P}(\mathcal{J}, f) = i(\bar{f}) \cdot i(f_b)$ by the hypothesis on the fixed point classes. The corollary now follows from the fact that the total space is a Jiang space and from a result of [1], which states that for any continuous

map $g: T^k \rightarrow T^k$, $N(g) = |L(g)|$. So, in particular, $i(f_b) = L(f_b)/N(f_b) = \pm 1$.

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