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1. Introduction

Throughout C(X) will denote the ring of all continuous real-valued functions on a Tychonoff space X, and C*(X) will denote the subring of bounded elements of C(X). The real line is denoted by R, and N denotes the (discrete) subspace of positive integers. A subset S of X such that the map $f \to f|_S$ is an epimorphism of C(X) (resp. C*(X)) is said to be *C-embedded* (resp. *C*-embedded*) in X. As is well-known, every $f \in C^*(X)$ has a unique continuous extension βf over its Stone-Cech compactification βX [GJ, Chapter 6]. That is, X is C*-embedded in βX.

In [NR], L. Nel and D. Riordan introduced the subset $C^{\#}(X)$ of C(X) consisting of all f such that for every maximal ideal M of C(X), there is an $r \in R$ such that $(f-r) \in M$, and they noted that $C^{\#}(X)$ is a subalgebra and sublattice of C(X) containing the constant functions. They show how $C^{\#}(X)$ determines a compactification of X in a number of cases and leave the impression that it always does. In [C1], E. Choo notes that this is true if X is locally compact and seems to conjecture that it need not be the case otherwise. In [SZ 1], O. Stefani and A. Zanardo show that every $f \in C^{\#}(R^{\omega})$ is a constant function, where R^{ω} denotes a countably infinite product of copies of R. In [SZ 2] they show that $C^{\#}(X)$

determines a compactification of X in case X is locally compact, pseudo compact, or zero-dimensional, and they describe the compactifications so determined when X is realcompact [GJ, Chapter 8].

In this paper, I show that under certain restrictions on X, the ring $C^{\#}(X)$ determines the Freudenthal compactification of X [I1, pp. 109-120], I observe that, at least in disguised form, $C^{\#}(X)$ has been considered by a number of authors other than those named above, and some conditions are given that are either necessary or sufficient for X to determine a compactification of X. In particular, it is shown that if X is realcompact, and $C^{\#}(X)$ determines a compactification of X, then X is rimcompact and it determines the Freudenthal compactification ΦX of X. There are realcompact rimcompact spaces X for which $C^{\#}(X)$ does not determine a compactification of X, but $C^{\#}(X)$ does determine ΦX if every point of x has either a compact neighborhood, or a base of open and closed neighborhoods. Other sufficient conditions are given for $C^{\#}(X)$ to determine ΦX. I close with some remarks and open problems.

2. Using $C^{\#}(X)$ to Compactify X

We will make use of the following characterization of $C^{\#}(X)$ due to a number of authors. Recall that $Z(f) = \{x \in X: f(x) = 0\}$ and υX denotes the Hewitt real compactification of X.

2.1 *Theorem. If* $f \in C(X)$, *then the following are equivalent.*

(a) $f \in C^{\#}(X)$.

(b) $f \in C^*(X)$ *and* $f[D]$ *is closed (and hence finite) for every* C-embedded *copy* D *of* N.

(c) $f \in C^*(X)$ *and* $f[Z]$ *is closed for every zero-set* Z *in* X.

(d) $f \in C^*(X)$ *and for every* $r \in R$, $\mathrm{Cl}_{\beta X} Z(f-r) = Z(\beta f-r)$.

(e) $f \in C^*(X)$ *and for every* $p \in \beta X \setminus \upsilon X$, *there is a neighborhood of* p *in* βX *on which* βf *is constant.*

The equivalance of (a) and (b) seems to appear first in [NR]. The equivalence of (a), (b), (c), (d) appears in [Cl], and that of (a), (b), (d), and (e) in [SZ 2]. Mappings that satisfy (d) are a special case of what are called WZ-maps by T. Isiwata, who showed that any map that sends zero-sets to closed sets in a WZ-map, and that a WZ-map on a normal space is closed [I 2], [W, p. 215]. More important for this paper is the following result. For any subset S of X, let $\mathrm{Fr}\, S = C\ell\, S \cap C\ell(X \setminus S)$ denote the *boundary* (or *frontier*) of S.

2.2 *Theorem. If* X *is realcompact and* $f \in C^{\#}(X)$, *then* $\mathrm{Fr}\, Z(f-r)$ *is compact for every* $r \in R$, *and* f *is a closed mapping.*

By Theorem 2.1 (d,e) if $r \in R$, then either $Z(f-r)$ is compact or $\mathrm{Fr}\, Z(\beta f-r) \subset X$. In the latter case, $\mathrm{Fr}\, Z(f-r) = \mathrm{Fr}\, Z(\beta f-r)$. In either case $\mathrm{Fr}\, Z(f-r)$ is compact. In [I.2, 1.3], T. Isiwata shows that a WZ-map with this latter property is closed, so the theorem is proved.

Recall that a space X is called *rimcompact* if it has a base of open sets with compact boundaries. X is said to be *zero-dimensional* at x if x has a base of neighborhoods with empty boundaries, and X is called *zero-dimensional* if it is zero-dimensional at each of its points. It is shown in [M3] that every rimcompact space has a compactification ΦX such that $\Phi X \backslash X$ is zero-dimensional, and wherever γX is a compactification of X with $\gamma X \backslash X$ zero-dimensional, there is a continuous map of ΦX onto γX leaving X pointwise fixed. ΦX is called the *Freudenthal compactification* of X.

In [D], R. Dickman shows that if X is rimcompact, then every $f \in C^*(X)$ such that $\mathrm{Fr}\, Z(f-r)$ is compact for every $r \in R$ has a (unique) extension in $C(\Phi X)$. Hence the following is an immediate consequence of Theorem 2.2.

2.3 *Corollary. If* X *is rimcompact and realcompact, then every* $f \in C^{\#}(X)$ *has a (unique) extension* $\Phi f \in C(\Phi X)$.

Suppose S is a subring of $C^*(X)$ that contains the constant functions and γX is a compactification of X such that every $f \in S$ has an extension $\gamma f \in C(\gamma X)$ and $S^{\gamma} = \{\gamma f\colon f \in S\}$ separates the points of γX. (That is if $x_1, x_2 \in \gamma X$ and $x_1 \neq x_2$, there is an $f \in S$ such that $\gamma f(x_1) = 0$ and $\gamma f(x_2) = 1$). Then by the Stone-Weierstrass Theorem, S^{γ} is dense in $C(\gamma X)$ in its uniform topology [GJ, 16.4], and we say that S *determines* the compactification γX of X. Note that S determines a compactification of X if points can be separated from disjoint closed sets by functions in S.

If $\gamma_1 X$ and $\gamma_2 X$ are compactifications of X for which there is a homeomorphism of $\gamma_1 X$ onto $\gamma_2 X$ keeping X pointwise fixed, then we write $\gamma_1 X = \gamma_2 X$.

For any space X, let $C_{\#}(\beta X) = \{\beta f\colon f \in C^{\#}(X)\}$ and note that $C_{\#}(\beta X)$ and $C^{\#}(X)$ are isomorphic. Similarly, if X is realcompact and rimcompact, then by Corollary 2.3, $C^{\#}(X)$ is isomorphic to $C_{\#}(\Phi X) = \{\Phi f\colon f \in C^{\#}(X)\}$.

A subring A of $C^*(X)$ is called *algebraic* if it contains the constant functions and those members $f \in C^*(X)$ such that $f^2 \in A$. If, in addition, A is closed under uniform convergence, then A is called an *analytic* subring of $C^*(X)$. The closure in the uniform topology of a subset B of $C^*(X)$ will be denoted by uB. It is noted in [GJ, 16.29], that if A is an algebraic subring of $C^*(X)$, then uA is an analytic subring.

If $B \subset C^*(X)$, then a *maximal stationary set* S of B is a subset of X maximal with respect to the property that every $f \in B$ is constant on S. In [GJ, 16.29-16.32], the following is established.

2.4 *If* X *is compact and* A *is an algebraic subring of* $C^*(X)$, *then every maximal stationary set of* A *is connected and* $uA = \{f \in A\colon f$ *is constant on every connected stationary set of* $A\}$.

If X is rimcompact and realcompact, then, by the above $C_{\#}(\Phi X)$ is an algebraic subring of $C^*(\Phi X)$. Next, I make use of the above to establish:

2.5 *Theorem. If* X *is a realcompact space and* $C^{\#}(X)$ *determines a compactification* γX *of* X, *then* X *is rimcompact and* $\gamma X = \Phi X$.

Proof. Suppose $x \in X$ and V is an open neighborhood of x. By assumption there is an $f \in C^{\#}(X)$ such that $f(x) = 0$ and $f(X\setminus V) = 1$. If $g = (f - \frac{1}{2}) \vee 0$, then, by Theorem 2.2 $Z(g)$ is a neighborhood of x with compact boundary that is contained in V. Hence X is rimcompact, and so $A = C_{\#}(\Phi X)$ is an algebraic subring of $C^*(\Phi X)$. Assume without loss of generality that X is not compact, let S denote a maximal stationary set of A, and suppose S has more than one point. Since A determines a compactification of X, it follows that $S \subset \Phi X \backslash X$. Since the remainder of X in ΦX is totally disconnected, S reduces to a point and Theorem 2.5 is established.

Next, I give an example to show that $C^{\#}(X)$ need not determine a compactification of a realcompact and rimcompact space. For any space X, let R(X) denote the set of points of X which fail to have a compact neighborhood. Clearly R(X) is closed since $X \backslash R(X)$ is open.

2.6 *Example. A realcompact rimcompact space* S *for which* R(X) *is a compact connected maximal stationary set.*

Let W* denote the space of ordinals that do not exceed the first uncountable ordinal ω_1, and let $W = W^* \backslash \{\omega_1\}$. It is well known that W* is compact and every $f \in C(W)$ is eventually constant [GJ, 5.13]. Let $X = [0,1] \times W^*$ with the topology obtained by adding to the product topology every subset of $[0,1] \times W$. Clearly X is rimcompact and $R(X) = [0,1] \times \{\omega_1\}$. Moreover, X is the union of a realcompact discrete space and the compact space R(X), so X is realcompact [GJ, 8.16]. Suppose $o \leq r < s \leq 1$ and $g \in C^*(X)$ is such that $g(r,\omega) \neq g(s,\omega)$. Since [0,1] is connected, since every $f \in C(W)$ is eventually constant, and since W has no countable cofinal subset, there is an $\alpha > \omega_1$, and an increasing sequence $\{x_n\}$ of real numbers between r and s such that $g(x_n,\alpha) \neq g(x_m,\alpha)$ if $n \neq m$. Thus g assumes infinitely many values on a closed discrete subspace of X and hence cannot be in $C^{\#}(X)$ by Theorem 2.1(b). So R(X) is a maximal stationary set of $C^{\#}(X)$.

It is clear that $C^{\#}(X)$ always contains both the subring $C_K(X)$ of all functions with compact support and the subring $C_F(X)$ of functions with finite range. Clearly any point of $X \setminus R(X)$ can be separated from any disjoint closed set by some element of $C_K(X)$, and if X is zero-dimensional at a point x, then x can be separated from any disjoint closed set by some element of $C_F(X)$. This together with 2.4 and Theorem 2.5 proves:

2.7 *Theorem. If* X *is a rimcompact, realcompact space that is zero-dimensional at each point of* R(X), *then* $C^{\#}(X)$ *determines* ΦX*; that is,* $u\, C_{\#}(\Phi X) = C(\Phi X)$.

Along these lines we have also:

2.8 *Theorem. If* X *is a rimcompact and realcompact space such that* $c\ell_{\Phi X}(\Phi X \setminus X)$ *is zero-dimensional, then* $u\, C_{\#}(\Phi X) = C(\Phi X)$.

Proof. By the remarks proceeding the proof of Theorem 2.7, if S is a maximal stationary set for $C_{\#}(\Phi X)$ with more than one point, then $S \subset c\ell_{\Phi X}(\Phi X \setminus X)$. Since the latter set is zero-dimensional, S reduces to a point and the conclusion follows.

In [Il, Theorem 36, p. 114], it is shown that if $\Phi X \setminus X$ is a Lindelöf space, then the Lebesgue dimension of $\Phi X \setminus X$ is zero. In [P, Corollary 5.8] it is shown that if F is a closed subset of a normal space Y, then the Lebesgue dimension of Y does not exceed the Lebesgue dimensions of A or $(Y\backslash A)$. It follows that if $R(X)$ is compact and zero-dimensional, then $c\ell_{\Phi X}(\Phi X\backslash X) = (\Phi X\backslash X) \cup R(X)$ is zero-dimensional, for these two motions of dimensionality coincide at 0 if X is compact; see [P, pp. 156-157]. Note also that $\Phi X\backslash X$ is a Lindelöf space if and only if every compact subset of X is contained in a compact subset with a countable base of neighborhoods; in which case we will say that X is of *countable type*. [I1, p. 119]. Thus we have established:

2.8 *Corollary. If* X *is a rimcompact, realcompact space of countable type, and* R(X) *is compact and zero-dimensional, then* $u\, C_{\#}(\Phi X) = C(\Phi X)$.

3. Remarks and Open Problems

A. In [N], the ring of all closed $f \in C(X)$ is considered for X locally compact and weakly *paracompact* (= *metacompact*). For X realcompact this latter ring coincides with $C^{\#}(X)$ by Theorem 2.2. Recall also that W. Moran showed in [M3] that if every closed discrete subspace of a normal metacompact space X is realcompact, then so is X. Also, examination of Example 3 of [N] shows that this latter need not hold if X fails to be normal.

B. In a private communication S. Willard notes that if $f \in C^*(X)$ and f is a closed mapping, then Z(f) has a countable base of neighborhoods in X. (I.e., $Z(f) = \bigcap_{i=1}^{\infty} f^{-1}(-1/i, 1/i)$). It would be of great interest to characterize the zero-sets of elements of $C^{\#}(X)$ at least in case X is rimcompact and realcompact. To determine which such spaces determine X, it would probably be enough to characterize zero-sets of restrictions to X of u $C_{\#}(\Phi X)$.

C. Willard notes also that if S is a countable subset of X and $c\ell_{\Phi X}S$ is connected, then S is a stationary set for $C^{\#}(X)$. It follows from a theorem of McCartney [M1, Proposition 3.12] that if $Y = [0,1] \times (0,1] \cup Z$, where $Z = \{(q,0): 0 \leq q \leq 1$ and q is rational$\}$, then $\Phi Y = [0,1] \times [0,1]$. Hence, by the latter remark of Willard cited above, Z is a stationary set for $C^{\#}(Y)$, so Y is a separable, metrizable rimcompact space such that $C^{\#}(Y)$ does not determine a compactification of Y.

D. Suppose $X = [0,1] \times Q \cap [0,1]$, where the open sets of X and those in the product topology together with any subset of $\{(a,b) \in X: b > 0\}$. Then $R(X) = \{(a,b) \in X: b = 0\}$ is compact and connected, X is rimcompact, realcompact, and determines ΦX. So the hypotheses of Theorem 2.7 or 2.8 are not necessary for X to determine ΦX.

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