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by

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CONTINUOUS LATTICES, TOPOLOGY AND TOPOLOGICAL ALGEBRA[1]

Karl Heinrich Hofmann

In 1974 Jean Dieudonné wrote an article in la Gazette des Mathématiciens in which he philosophized on what he believed to constitute noble mathematics and servile mathematics. He arranged all the mathematics he could think of in a vertical hierarchy. General topology, located on the lowest level of the hierarchy appears to be a rather ignoble commodity. And where is lattice theory? Must we conclude from its absence that, having been banned from the hierarchy, it is totally without redeeming value?

Nevertheless, this lecture will be on topology and lattice theory. We will discuss under the name of continuous lattices a relatively novel class of lattices which emerged quite independently in various guises in different branches of mathematics ranging from computer science to category theory under the dictate of need. Recent research demonstrated its significance and elucidated its role as a link between different areas. For the purpose of this lecture we will restrict our attention primarily to bonds with topology.

Even Dieudonné acknowledged that any line of mathematics may escalate to a higher level of existence in his mathematical universe under the influence and imagination of a

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new leader. Our story will be about the leadership of mathematicians like J. D. Lawson and Dana Scott.

Introduction

In order to define a continuous lattice, we observe that in any partially ordered set we may define an auxiliary transitive relation by $x << y$ whenever for any subset A with $y \leq \sup A$ we find a finite subset $F \subseteq A$ with $x \leq \sup F$. The relation $x << y$ implies $x \leq y$, but it may never hold for all we know; if there is a smallest element 0, then $0 << x$ for all x. More generally, for a finite partially ordered set we have $x << y$ iff $x \leq y$. Otherwise, even in cases where this relation is prevalent, it is not generally reflexive. An element x with $x << x$ is called *compact*.

Now we can introduce the following definition

Definition. A partially ordered set L is called a *continuous lattice* iff

(0) L is a complete lattice (i.e. every subset has a least upper bound)

(1) $(\forall x \in L)\ x = \sup\{u \in L: u << x\}$.

It is called an *algebraic lattice* iff it satisfies (1) and

(2) $(\forall x \in L)\ x = \sup\{k \in L: k << k \leq x\}$.

The subset K(L) of compact elements is always a sup-semilattice.

A lattice is *arithmetic*, iff it is algebraic and the set K(L) is a sublattice.

The implications finite $\Rightarrow$ arithmetic $\Rightarrow$ algebraic $\Rightarrow$ continuous $\Rightarrow$ complete are clear, and none of these implications can be reversed. We will see much of most of these

in the sequel; each class is closed under forming arbitrary products. Each complete chain is a continuous lattice. The lattice of open sets of a locally compact space is continuous, where $U << V$ if we have a compact set with $U \subseteq C \subseteq V$. The lattice of open sets of a Boolean space is algebraic with the compact open sets being precisely the compact elements in the lattice. Scott's universe $P\omega = \{x | x \subseteq \omega\}$ for recursive function theory and the λ-calculus is an algebraic lattice. The lattices of all ideals of a ring, or of subgroups of a group, or of congruences of a general algebra are algebraic, the compact elements being the finitely generated objects. (So much for terminology.)

The lattice of two sided *closed* ideals of a C*-algebra is continuous. The lattice of lower semicontinuous extended functions $f: X \to \mathbf{R} \cup \{\pm\infty\}$ on a compact space is continuous. The concepts of algebraic and arithmetic lattices are traditional; the concept of a continuous lattice has been introduced by D. Scott about 5 years ago and he baptized them. They should not be mixed up with the continuous geometries of von Neumann. The definition given above is about two years old. Let me draw your attention to one feature which sets these concepts apart from the cherished features of classical lattice theory: They are asymmetrical insofar as conditions (1) and (2) are not symmetric (i.e. invariant under passage to the new relation $x \leq^{op} y$ iff $y \leq x$. In fact $<<^{op}$ is not derived from $\leq^{op}$ as is $<<$ from $\leq$. We will return to this motive of asymmetry time and again; it may be one of the reasons that the fruitful exploitation of these concepts is recent.

Some General Thoughts on the Relation Between Topology and General Topology

Traditionally, topology and lattice theory touch in several contexts. We list a few:

1) To each topological space X one associates the lattice O(X) of all open sets (sometimes the sublattice OC(X) of open closed subsets). There are ways to associate with a lattice L a topological space called its spectrum Spec L. One finds natural functions $X \to \text{Spec } O(X)$ and $L \to O(\text{Spec } X)$ which may be interpreted and utilized in different ways: Firstly, $X \to \text{Spec } O(X)$ will reveal how much information on the space X can be retrieved from knowing the lattice O(X) and to what extent "topology without points" is possible. The history of this line of investigation bears the marks of Dowker, Papert and Papert, Bénabou, Isbell. Secondly, $L \to O(\text{Spec } X)$ affords a method to represent lattices as lattices of open sets. One would not expect a universally applicable representation theory since O(X) is always a distributive lattice whereas many important lattices (such as the lattice of subgroups of a group, or the lattice of projections of a Hilbert space (playing an important role in the foundation of quantum mechanics)) are not distributive. In many respects distributivity is to the lattice theoretician what commutativity is to the group theoretician: There are many interesting problems attached to the special class, but others are entirely outside its scope. The usefulness of the representation theory of lattices through this method has the most venerable tradition in lattice theory, as is

illustrated by the classical Boole-Stone representation theorem for Boolean lattices, which was expanded by Stone to a wide class of distributive lattices.

2) On each space we have a quasi order given by $x \leq y$ iff $y \in \{x\}^-$ (iff for all open sets U one has $x \in U$ if $y \in U$). This relation is a partial order iff the space is T_o. Conversely, if a partial ordered set is given, there are numerous ways of making a T_o-topology from the order. (Example: Basic open sets: $\{x \mid \alpha \not\leq x\}$, $\alpha \in L$).

Why would T_o-spaces deserve particular attention when so many articles or lectures begin "we will assume throughout that all spaces are Hausdorff" and when, for instance in the theory of topological groups, T_o-separation already implies Hausdorff complete regularity?

One principal motivation for considering T_o-spaces is the spectral theory of commutative rings (algebraic geometry) and of algebras such as Banach algebras, operator algebras (functional analysis), or the study of congruences on algebras of a general type (universal algebra). Specifically, let R be a commutative ring, and let Spec R be the set of all prime ideals P (i.e. R\P is a multiplicative semigroup), and call a subset $h(X) = \{P \in \text{Spec } R: X \subseteq P\}$ the hull of $X \subseteq R$. Trivally, $h(\cup_j X_j) = \cap_j h(X_j)$; but the primeness of P guarantees also that $P \in h(x) \cup h(y)$ iff $x \in P$ or $y \in P$ iff $xy \in P$ iff $P \in h(xy)$. The sets $\sigma(x) = \text{Spec } R \backslash h(x)$ thus form a basis of open sets of a topology, called the *hull-kernel topology*. It is at the heart of all spectral theory. The hull kernel topology is T_o, but rarely better. Notice that the sets $\sigma(x)$ are quasicompact and open.

If A is a C*-algebra, then the set $\hat{A}$ of unitary equivalence classes of irreducible *-representations on a Hilbert space is a non-separated topological space with the property that the lattice Id A of closed two sided ideals of A is isomorphic to $O(\hat{A})$.

If A is a universal algebra, belonging to an equational class, then the lattice of congruences is a central object of study in universal algebra. One may again assign to the algebra a space, namely, the space of irreducible congruences. Its topology is a hull-kernel topology. Again most spaces arising in this way are not separated. Another reason to consider T_0-spaces comes from classical considerations of real valued functions. The study of lower semicontinuous functions $f: X \to \mathbf{R} \cup \{\pm\infty\}$ is subsumed in continuous function space theory if we admit on the range the T_0-topology of all open intervals which are unbounded above.

3) A third motive relating topology and lattices arises from the endeavor to introduce on a lattice in a natural way topologies relative to which the operations $(x,y) \longmapsto xy = x \wedge y$, $x \vee y$ are continuous. Much has been written on this branch of topological algebra. Any totally ordered set can serve as motivating example with the interval topology.

1. T_0-Spaces and Continuous Lattices

Let us introduce some notation: For a poset L and a subset $A \subseteq L$ we write $\uparrow A = \{x \in L: (\exists a)\ a \in A \text{ and } a \leq x\}$. We write $\uparrow a$ in place of $\uparrow\{a\}$. The notation $\downarrow A$ is defined dually. We write $\mathop{\downarrow}\limits_{\uparrow} x = \{y \in L \mid y << x\}$ and $\mathop{\uparrow}\limits_{\downarrow} x = \{y \in L \mid x << y\}$.

1.1. *Definition.* Let L be a poset. Then $U \subseteq L$ is *Scott-open*, iff $\uparrow U = U$ and $(\forall D)$ D up-directed with $\sup D \subseteq U \Rightarrow D \cap U \neq \emptyset$. The collection of all Scott-open subsets is the *Scott-topology*. Every $L\backslash\uparrow x$ is Scott-open (whence the Scott topology is T_0). The topology generated by all $L\backslash\uparrow x$, $x \in L$ is called the *spectral topology*. It is likewise T_0. We will see how it relates to hull-kernel topologies. The least upper bound of the two topologies will be called the *Lawson topology*.

Notice that the spectral topology is not the opposite of the Scott topology. Thus the Lawson topology is not a symmetric concept. Every Scott-open set U is Lawson-open, and by 1.1, the converse is true if $U = \uparrow U$ holds; thus a set U with $\uparrow U = U$ in a complete lattice is Scott open iff it is Lawson open. We note that $y \in \text{int} \uparrow x$ (with int denoting interior relative to the Scott- or the Lawson-topology) implies $x << y$. The converse holds in continuous lattices.

We now have the following theorem:

1.2. *Theorem* (1977). *Let* L *be a complete lattice. Then the Lawson-topology on* L *is quasicompact* T_1 *and the second of the following statements implies the first*

(1) The Lawson-topology is Hausdorff.

(2) L *is a continuous lattice.*

Proof. The subbasic sets of the Lawson topology are of the form U with a Scott open U and $L\backslash\uparrow x$, $x \in L$. Let $L \subseteq \cup_i U_i \cup \cup_j (L\backslash\uparrow x_j) = \cup_i U_i \cup (L\backslash\cap_j \uparrow x_j)$. Suppose no finite subfamily covers. For any finite set J of indices j, let $x_J = \sup_J x_j$. Then for all i and all J there is an s with

$s \notin U_i$ and $s \in \uparrow x_J$. Thus $x_J \notin U_i$. If $x = \sup x_J = \sup x_j$, then $x \notin L\setminus\uparrow\sup x_j = \bigcup_j (L\setminus\uparrow x_j)$. Hence there is an i with $x \in U_i$. By the definition of the Scott topology, there is a J with $x_J \in U_j$, and this is a contradiction. Thus the Lawson topology is quasicompact. If we have $x \not\leq y$, then $L\setminus\uparrow x$ is a neighborhood of y not containing x and $L\setminus\uparrow y$ is a neighborhood of x not containing y. Thus the Lawson-topology is T_1.

(2) $\Rightarrow$ (1): Let $x \not\leq y$; by (2) find $u \ll y$ and $x \not\leq u$. Then $L\setminus\uparrow u$ and $\uparrow u$ are disjoint neighborhoods of x and y, respectively.

Remark. If L is meet continuous (s.1.6 below), then (1) <=> (2). Indeed if $x < y$, then by (1) there is a basic neighborhood $U\setminus\uparrow F$ of x, U Scott-open and F finite, such that y is in the interior of its complement, i.e. $y \in \operatorname{int} \uparrow F$. By meet continuity, $\operatorname{int} \uparrow F = \bigcup\{\uparrow u : u \in F\}$. Hence $u \ll y$ for some $u \in F$, and $u \not\leq x$. The case $y \not\leq x$ is reduced to this one. Hence $y = \sup \downarrow y$.

This is a first characterization of continuous lattices among complete lattices by topological means. The first part of the proof is due to D. Scott.

We noticed that on any posets there exist T_0-topologies. Let us now start from T_0-spaces and detect under what conditions one returns to lattices.

All spaces are T_0 from here on out. We present Scott's approach (1972).

1.3. *Definition.* A space Z is called *injective* iff for every pair $X \subseteq Y$ of spaces every continuous function $f: X \to Z$ extends to a continuous function $F: Y \to Z$.

This is equivalent to saying that Z is a retract of every containing space.

Note. This definition is categorical in the following sense: Let $\mathcal{A}$ be a category and $\mathcal{J}$ a class of monics. Then $Z \in \text{ob } \mathcal{A}$ is a *$\mathcal{J}$-injective* iff for every pair of morphisms $j: X \to Y$ and $f: X \to Z$ with $j \in \mathcal{J}$ there is a morphism $F: Y \to Z$ with $f = Fj$. If $\mathcal{A} = T_0$ is the category of T_0-spaces and $\mathcal{J}$ the class of embeddings, then the injectives of 1.3 are precisely the $\mathcal{J}$-injectives.

Examples. Trivially every singleton space is injective. The lattice $2 = \{0,1\}$ with the Scott topology has $\{1\}$ as the only non-trivial open set (the "Sierpinsky space"); it is easily seen to be injective. Indeed a continuous function $f: X \to 2$ is just the characteristic function of an open set and thus, by the definition of induced topology, extends.

Products of injectives are injectives, as are retracts. Since every T_0-space X is embeddable into $2^{\text{Top}(X,2)}$, every space is embeddable into an injective one and the retracts of the spaces 2^M are precisely the injective ones.

Here then is Scott's theorem:

1.4. *Theorem* (Scott 1972). *Let* L *be a* T_0*-space with the partial relation* $\leq$ *defined by "*$x \leq y$ *iff for all open sets* U *we have* $x \in U$ *implies* $y \in U$*."*

Then the following two conditions are equivalent:

(1) L *is an injective* T_0*-space.*

(2) L *is a continuous lattice (relative to* $\leq$*).*

Moreover, if these conditions are satisfied, then the topology on L *is the Scott-topology associated with* $(L,\leq)$.

In order to speak of functions in greater detail, we denote the full subcategory of the category Top of topological spaces and *continuous* maps whose objects are injective T_0-spaces (hence *continuous* lattices) by Cont. One of Scott's main discoveries was that Cont is a cartesian closed category. This does not only allow the creation of new continuous lattices by forming products, but also by forming function spaces which was one of the main motivations for Scott in his creating models for the λ-calculus. (We will speak on the formation of quotients and subobjects later when we have a more subtle understanding of special types of morphisms.)

1.5. *Lemma. A function* $f: L \to L'$ *between two complete lattices is Scott-continuous iff it preserves sups of up-directed sets. A function* $f: L_1 \times L_2 \to L$ *of several variables (with continuous lattices* L_1, L_2 *and* L*) is continuous with the product of the Scott-topologies iff it is continuous in each variable separately.* (D. Scott).

The result about separate and joint continuity of multivariable functions is somewhat delicate. It seems unknown whether it is true for complete lattices in general. After a recent observation by Hofmann and Gierz it suffices that L_1, L_2 are complete and the Scott topology $O(L_2)$ of L_2 is a continuous lattice.

The sup operation $(x,y) \longmapsto x \vee y: L \times L \to L$ is Scott continuous in any complete lattice. (This does not say that it is continuous for the product topology on $L \times L$ which is possibly coarser than the Scott topology on $L \times L$. For both topologies to be equal it suffices that O(L) be continuous.)

This fails for the inf operation in general. Hence one is challenged to add a definition:

1.6. *Definition.* A lattice is called *meet continuous* (MC) iff it is complete and $(x,y) \longmapsto xy: L \times L \to L$ is continuous. This latter condition is equivalent to the following condition:

(MC) For each chain $D \subseteq L$ and each $x \in L$ we have $x \sup D = \sup xD$.

Recall that a complete lattice is *Brouwerian* iff it satisfies (BC): For each subset $D \subseteq L$ and each $x \in L$ we have $x \sup D = \sup xD$. Thus every Brouwerian complete lattice is MC, and a meet continuous lattice is Brouwerian iff it is distributive. In particular, all lattices $O(X)$ are Brouwerian, hence MC.

From our later observations it will be immediate (and Scott showed a direct proof) that every continuous lattice is meet continuous; the converse is not true. Thus we have refined the hierarchy of lattices which we gave in the beginning. We also observe as a consequence of 1.2 and 1.6

1.7. *Proposition. If* L *is a meet continuous lattice, then* L *is a quasicompact* T_1*-topological semilattice (relative to inf and the Lawson-topology), provided* $O(L)$ *is continucus.*

Proof. Let $m: L \times L \to L$ be the multiplication $(x,y) \longmapsto xy$. If $U \subseteq L$ is Scott open, then $m^{-1}(U)$ is open since L is meet continuous. If $x \in L$ is arbitrary, then $m^{-1}(L \setminus \uparrow x) = L \times L \setminus \{(s,t) \in L \times L \mid st \geq x\} = L \times L \setminus \uparrow(x,x)$ is spectral-open. ($O(L)$ continuous $\Rightarrow O(L \times L) = O(L) \times O(L)$.)

From 1.7 and 1.2 we draw the corollary that a continuous lattice is a compact topological semilattice in the Lawson-topology.

If X is a topological space, then O(X) is meet continuous. If O(X) is a continuous lattice, then O(X) is a compact T_2 topological $\cap$-semilattice, and it is a topological lattice for the Scott topology. As Isbell would say "any continuous topology is a topological topology," which makes more sense than "a rose is a rose is a rose." He points out, too, that contrary to what had been previously asserted by Scott, himself and myself, a topology need not be topological in general.

If X is an injective T_0-space, then the lattice O(X) of Scott open even subsets is isomorphic to Cont(X,2) (where 2 has the Scott topology). In the process of establishing that Cont is cartesian closed, Scott showed:

1.8. *Lemma. If* L,L' $\in$ Cont, *then the topology of pointwise convergence is the Scott-topology for the pointwise partial order (*f $\leq$ g *iff* f(x) $\leq$ g(x)*) and* Cont(L,L') $\in$ Cont *(i.e.* Cont(L,L') *is a continuous lattice).*

In particular, then if X $\in$ Cont, then O(X) $\cong$ Cont(L,2) is a continuous lattice. We notice:

1.9. *Lemma. For a topological space* X *the following statements are equivalent:*

(1) O(X) *is a continuous lattice.*

(2) For each point x *and each open neighborhood* U *of* x *there is an open neighborhood* V $\subseteq$ U *of* x *such that*

every filter on V *clusters in* U *(i.e. every ultrafilter on* V *converges in* U*).*

Isbell calls these spaces *semi-locally bounded*, while A. S. Ward called them *quasi locally compact*. For the present purposes I will adhere to the following less colorful definition:

1.10. *Definition*. A space will be called a CL-*space* iff O(X) is a continuous lattice.

We have seen that all injective T_0-spaces are CL-spaces. From 1.6 it is clear (and B. J. Day and G. M. Kelly are on record (1970) for this observation in a different terminology) that every locally quasicompact space is a CL-space (where a space is *locally quasicompact* if every point has a basis of quasicompact neighborhoods). We will say more on this later. But in the context of function spaces we are now ready for Isbell's theorem which is the conclusive word on function spaces and continuous lattices:

1.11. *Theorem*. (Isbell 1975) *Let* L *be a complete lattice and* X *a space, and consider* L *with its Scott topology (making it into a* T_0*-space). Consider* S = Top(X,L) *with the pointwise partial order. Then the following statements are equivalent:*

(1) X *is a* CL*-space, and* L *a continuous lattice.*

(2) S *is a continuous lattice (i.e. an object of* Cont *in its Scott topology which is the topology of pointwise convergence).*

Proof. If (1) is satisfied, then $\underline{Top}(X,2) \cong O(X)$ is a continuous lattice. Now L is a retract of some 2^M, but $\underline{Top}(X,-)$ preserves products and retracts. Thus $\underline{Top}(X,L)$ is a retract of an injective space, hence is injective. So (2) holds. Conversely, suppose (2). Consider the retraction $f \longmapsto \sup f: \underline{Top}(X,L) \to L$ which is right adjoint to const: $L \to \underline{Top}(X,L)$, $\text{const}(s)(x) = s$: Indeed $s \geq \sup f$ iff $\text{const } s \geq f$. But const preserves sups, hence is continuous. Thus L is a retract of $\underline{Top}(X,L)$ hence is a continuous lattice. Now 2 is a retract in $\underline{Cont}$ of L, hence $\underline{Top}(X,2)$ is a retract in $\underline{Top}$ of $\underline{Top}(X,L)$, hence is injective and so X is a CL-space.

We notice that the augmented reals $\tilde{\mathbf{R}} = \tilde{\mathbf{R}} \cup \{\pm\infty\}$ form a continuous lattice and that for any locally compact space X, the space $\underline{Top}(X,(\tilde{\mathbf{R}}, \text{Scott}))$ is exactly the space $LC(X, \tilde{\mathbf{R}})$ of extended lower semicontinuous functions. Isbell's Theorem (1.11) together with 1.2 then says that there is a compact function space topology on $LC(X, \tilde{\mathbf{R}})$ relative to which inf is a continuous operation. I do not know whether this fact is known in classical analysis.

We mentioned the observation by Day and Kelly that locally quasicompact spaces X have a continuous lattice of open sets. A complete characterization of CL-spaces appears to be difficult after Isbell and Hofmann and Lawson found CL-spaces in which every quasicompact set has empty interior. Nevertheless something can be said. Let us recall that a space is primal (or sober) if every closed set which is irreducible (i.e. cannot be the union of two proper subsets) has a dense point. Every T_o-space allows a universal

embedding $X \to \check{X}$ into a sober space. We now have the following theorem.

1.12. *Theorem* (Hofmann and Lawson, 1977). *For a* T_0*-space* X *the following statements are equivalent:*

(1) X *is a* CL*-space.*

(2) The sobrification of X *is locally quasicompact.*

Let L be a complete lattice and let Spec L be the set of all primes $p < 1$ with the topology induced by the spectral topology; this topology is also called the hull-kernel topology. For instance, $\check{X}$ = Spec O(X), and the embedding of X is given by $x \longmapsto X\backslash\overline{\{x\}}$. Let us call a subset X of a lattice L *order generating* if every element is the inf of a subset of X. We then have

1.13. *Theorem* (Hofmann and Lawson, 1977). *Let* L *be a distributive continuous lattice. Then* Spec L *is order generating and the map* $x \longmapsto$ Spec $L\backslash\uparrow x$: $L \to O(\text{Spec } L)$ *is an isomorphism of lattices. Further,* Spec L *is a sober locally quasicompact space, and all such spaces are so obtained.*

More specifically, any order generating subspace X *of* Spec L *is a* CL*-space, and every* CL*-space is so obtained.*

2. Compact Semigroups

Workers in topological algebra and functional analysis are interested in compact topological and semitopological semigroups. Let us accept for the moment that the abelian ones play a particularly important role. Suppose that S is a monoid (= semigroup with identity). Then $E(S) = \{s \in S \mid s^2 = s\}$ is a submonoid and if S is topological then

E(S) is closed (if S is semitopological, this is not generally the case). In the structure theory of S the monoid E(S) is an important functorial invariant. Let us mention an example even though it does not totally fit the topological case: If G is a locally compact abelian group and M(G) its measure algebra, then its structure space $\Delta M(G)$ according to J. Taylor is a compact semitopological abelian semigroup in which certain elements of $E(\Delta M(G))$ (so called critical idempotents) correspond bijectively to locally compact group topologies on G refining the given one and they form the index set for an explicit sum representation of the cohomology of $\Delta M(G)$. In fact $\Delta M(G)$ is the character semigroup of a compact abelian topological semigroup.

2.1. *Definition.* A *compact topological semilattice* is a compact space with a continuous commutative idempotent multiplication with identity.

Apart from the applications, compact topological semilattices are, in the simplest cases, easy to come by and therefore yield a useful supply of examples in the theory of compact semigroups. We will always use the word compact semilattice for compact commutative idempotent monoid. Every product of compact semilattices is one, every closed subsemilattice of a semilattice is again one. Quotients of compact semilattices are compact semilattices, so they form a quasivariety. All totally ordered order complete sets (such as the unit interval I and the Cantor set C) are compact semilattices under min. By the preceding they co-generate a large subvariety if not (as would be a priori conceivable

and was indeed believed for some years) the entire class. It was observed early that compact semilattices exhibit some interesting phenomena which are not present with compact groups: The Cantor semilattice C allows I as quotient (via Cantor's gap closing function) but dim C = 0 and dim I = 1 (by anybody's dimension count), and so there are dimension raising morphisms around. This was known to Wallace and Koch around 1960 but was only recently clarified completely (Hofmann, Mislove, Stralka 1973). Studies in compact topological lattices and semilattices were undertaken by Wallace's school since the fifties.

We mention one basic theorem due to R. J. Koch. Let us write $\underline{CS}$ for the category of compact semilattices with continuous semilattice morphisms preserving identities.

2.2. *Theorem. Let* $S \in \underline{CS}$. *Then the following statements are equivalent:*

(1) S *is connected.*

(2) If $k \in S$ *is a local minimum (i.e.* k *is isolated in* Sk*) then* $k = 0$.

(3) For each $s \in S$ *there is a connected compact chain* $T \subseteq Ss$ *with* $\{s,0\} \subseteq T$.

(4) S *is acyclic (over any coefficient module).*

The topological algebra of semilattices began moving strongly when J. D. Lawson arrived and attacked the question whether or not compact semilattices had a character theory somewhat like that of compact abelian groups, where morphisms into the circle group were replaced by morphisms into the semilattice I. The first problem which poses itself in this

context is whether or not the morphisms $S \to I$ separate the points of a compact semilattice. This is tantamount to asking whether any compact semilattice is a subsemilattice of some "cube." Lawson proved the following theorem:

2.3. *Theorem. Let* $S \in \underline{CS}$, *the following statements are equivalent:*

(1) $\underline{CS}(S,I)$ *separates the points of* S.

(2) [*resp. (2')*] S *has small* [*open*] *semilattices (i.e. every point has a basis of* [*open*] *neighborhoods* U *with* $U^2 \subseteq U$*).*

(3) S *is ultra-uniform, i.e. the uniform structure of* S *has a basis of neighborhoods* W *of the diagonal in* $S \times S$ *which is a subsemilattice of* $S \times S$.

It was believed for some time that all $\underline{CS}$-objects satisfied these conditions. But Lawson established the existence of a topologically one dimensional semilattice on which all morphisms into the unit interval are constant. That was in 1970, and it is hard even today to come by examples of $\underline{CS}$-objects which do not satisfy the conditions of Theorem 2.2. The recent discovery by J. W. Roberts of compact convex subsets without extreme points in topological vector spaces yields a class of such examples, since for every compact convex subset X of a topological vector space the set S of all compact convex subsets $A \subseteq X$ is a compact semilattice under the operation $(A,B) \longmapsto AB$ = closed convex hull of $A \cup B$ (R. A. Jamison, J. D. Lawson). The semilattice satisfying the conditions of 2.2 are called *compact Lawson semilattices*. Let us denote the full subcategory of $\underline{CS}$ consisting

of compact Lawson semilattices by $\underline{CL}$. By Lawson's discovery there is a gap between $\underline{CS}$ and $\underline{CL}$. One wonders about the category of all compact semitopological semilattices (in which multiplication if only separately continuous). But here we have a more recent result of Lawson's:

2.4. *Theorem* (1976). *Any compact semitopological semilattice is topological.*

The following question poses itself naturally:

Let $S \in \underline{CS}$; what are topological (or algebraic) hypotheses on S which entail $S \in \underline{CL}$? Lawson showed that $\dim S < \infty$ and local connectivity are enough. In fact he proves more. We need a few ideas on the connectivity relation on spaces. Let C_X be the relation of connectivity on the space X, i.e. $x\ C_X\ y$ iff there is a connected subspace $Y \subseteq X$ with $x,y \in Y$. We say that a subspace $Y \subseteq X$ is *fitted into* X if $C_Y = C_X | Y$ $(= C_X \cap (Y \times Y))$. We say that an equivalence relation R on a space is lower semicontinuous if $R(A)^- = R(R(A)^-)$ for all $A \subseteq X$. Finally we say that a neighborhood W of x in X is *well-fitted* iff (i) W is compact, (ii) each point $v \in \text{int } W$ has a basis of compact neighborhoods V such that V is fitted into W and (iii) C_W is lower semicontinuous.

We notice that every space which is locally a product of a compact zero dimensional space and a locally connected space has a well-fitted neighborhood around each point.

2.5. *Theorem* (Lawson 1977). *If* $S \in \underline{SC}$ *has a well-fitted finite dimensional neighborhood at each point, then* $S \in \underline{CL}$.

In particular, one retrieves the old fact, due to a theorem of Numakura's:

2.6. *Remark. If* $S \in \underline{CS}$ *is zero dimensional, then* $S \in \underline{CL}$.

The full category of zero dimensional objects in $\underline{CS}$ is called $\underline{Z}$.

While character theory did not work too well for $\underline{CS}$, and not particularly well for $\underline{CL}$, it works superbly for $\underline{Z}$. In fact we have:

2.7. *Theorem. The category* $\underline{Z}$ *is dual to the category of semilattices* $\underline{S}$ *under the functors* $S \longmapsto \underline{CS}(S,2): \underline{Z} \to \underline{S}$ *and* $T \longmapsto \underline{S}(T,2): \underline{S} \to \underline{Z}$, *where the hom sets are given the pointwise structures.*

This is a parallel to Pontryagin duality for compact abelian groups; it is due to various authors such as Austin, Schnepermann, Bowman and was investigated in detail by Hofmann, Mislove, Stralka.

What has all of this got to do with lattice theory? A very primitive observation is that the partial order given on a compact semilattice S by $x \leq y$ iff $xy = x$ gives S the structure of a complete lattice. In fact it is not hard to see that we obtain a meet continuous lattice. If we denote with $\underline{MC}$ the category of all meet continuous lattices with morphisms preserving arbitrary infs and sups of updirected sets (i.e. Lawson-continuous morphisms) then we have $\underline{CS} \subseteq \underline{MC}$ (considering any obvious forgetful functor an inclusion). The question now arises: Is every $\underline{CS}$-object in fact a

continuous lattice? The answer is no and the precise circumstances are exactly known by a result first established by Hofmann and Stralka (1974).

2.8. *Theorem. Let* L *be a complete lattice. Then the following statements are equivalent:*

(1) L *is a continuous lattice.*

(2) There is on L *a compact Hausdorff topology such that* L *is a compact Lawson semilattice.*

If these conditions are satisfied, then there is only one such topology, the Lawson-topology.

We add to this theorem our earlier observations on morphisms:

Let f: L → L' *be a function between two lattices. Then the following statements are equivalent:*

(I) f *is a morphism of compact Lawson semigroups (relative to the Lawson-topologies).*

(II) f *is a morphism of continuous lattices (as specified after 1.14).*

Notice that the appearance of a semilattice in (2) is evidence once more of an asymmetric condition of work on the lattice L.

Since every continuous lattice has exactly one compact semilattice structure (which indeed makes it into a compact Lawson semilattice) and since the underlying lattice structure of a compact Lawson semilattice is that of a continuous lattice, the categories of continuous lattice and of compact

Lawson semilattices are literally equal (not only naturally isomorphic). Our nomenclature is therefore justified: $\underline{CL}$ reads equally well as "continuous lattices" and as "compact Lawsons." We now have the following hierarchy of categories of complete lattices: $\underline{CL} \subseteq \underline{CS} \subseteq \underline{MC}$. As we saw, very little has been known on $\underline{CS} \setminus \underline{CL}$, and the same is true for $MC \setminus CS$.

The most recent result in this direction emerges from contributions by Gierz, Hofmann, Lawson, Mislove and reads as follows:

2.9. *Theorem* (1977). *Let* L *be a meet continuous lattice. Then the following are equivalent:*

(1) L *carries a (unique) compact semilattice topology (i.e.* $L \in \underline{CS}$*)*

(2) O(L) *(the lattice of Scott-open sets) is a continuous lattice (i.e.* $O(L) \in \underline{CL}$*)*

(3) In the Scott topology, L *is a locally quasicompact space.*

Furthermore, the following are equivalent:

(I) L *carries a (unique) compact Lawson semilattice topology (i.e.* $L \in \underline{CL}$*)*

(II) $O(L) \in \underline{CL}$ *and every element of* O(L) *is a sup of coprimes*

(III) In the Scott topology, L *is a locally quasicompact space with a basis of open filters.*

We still know little on sufficient conditions which would allow us to introduce on an $\underline{MC}$ lattice a $\underline{CS}$ or a $\underline{CL}$ topology.

The only thing we have so far is a result of Lawson's (1976):

2.10. *Theorem. If* $S \in \underline{MC}$, *and if* S *does not contain a copy of a free semilattice in infinitely many generators, then* $S \in \underline{CL}$.

This is just about all we know about what happens above $\underline{CL}$. And below? Here the situation is quite good. In fact we have the following result due to Hofmann, Mislove and Stralka, which historically preceded Theorem 2.8:

2.11. *Theorem. Let* L *be a complete lattice. Then the following statements are equivalent:*

(1) L *is an algebraic lattice.*

(2) There is on L *a compact zero dimensional Hausdorff topology such that* L *is a compact semilattice.*

Thus the full subcategory of algebraic lattices in $\underline{CL}$ is precisely $\underline{Z}$, the category of compact zero dimensional semilattices. More information can be given on the Pontryagin duality of $\underline{Z}$ and $\underline{S}$ first mentioned in Theorem 2.7.

2.12. *Proposition. Let* $L \in \underline{Z}$. *Then the character semilattice* $\hat{L} = \underline{CL}(L,2)$ *of* L *is isomorphic to the sup-semilattice* K(L) *of compact elements of* L *under the map* $\chi \longmapsto \min \chi^{-1}(1) : \hat{L} \to K(L)$. *An element* $s \in L$ *is in* K(L) *iff it is a local minimum in the Lawson-topology or the Scott topology.*

Let S be a semilattice written multiplicatively. Then the character semilattice $\hat{S} = \underline{S}(S,2)$ is isomorphic to the semilattice $\mathcal{F}(S)$ of filters on S under the map $\chi \longmapsto \chi^{-1}(1) : \hat{S} \to \mathcal{F}(S)$.

There are functorial relations in the hierarchy of lattices which point in the vertical direction. Thus it is often useful to associate with a complete lattice L a $\underline{Z}$-object PL which is simply the set of lattice ideals of L with the containment as order. The compact elements are the principal ideals and there is a canonical map r_L: PL → L given by $r_L(J) = \sup J$. Since r_L is right adjoint to the map $x \longmapsto {\downarrow}x$, it preserves arbitrary sups. We have the following theorem (Hofmann and Stralka 1974):

2.13. *Proposition. Let* L *be a complete lattice and* r_L: PL → L *the* SUP-map *given by* $r_L(J) = \sup J$. *Then*

(a) r_L *is a lattice morphism iff* L ∈ $\underline{MC}$.

(b) r_L *preserves arbitrary infs (in addition to arbitrary sups) iff* L ∈ $\underline{CL}$.

For a compact Lawson semilattice S the function r_S: PS → S is a canonical way of representing S as a quotient of a compact zero dimensional semilattice. Notice that the proposition is yet another characterization of continuous lattices. If L ∈ $\underline{CL}$ then the right adjoint of r_L associates with x ∈ L the unique smallest dense ideal ${\downarrow\!\!\downarrow}x$ in ${\downarrow}x$.

Conclusion

Only the relations between topology and topological algebra were discussed in this lecture, and those only in a fleeting way. We could not touch the emergence of continuous lattices from the theory of computing which is in fact one of their sources opened up by Dana Scott. In category theory Alan Day and O. Wyler discovered that continuous

lattices are precisely the monadic algebras of filter and ultrafilter monads; this provides the link between continuous lattice theory and universal algebra. In recent months numerous occurrences of continuous lattices were discovered in functional analysis, and in fact R. Giles has noticed their appearance in his work in formalizing interpretive rules in the foundation of physics.

Research in the area is in full flux, and it would be too premature to propose a summary at this time. It may, however, be of use to present a bibliography of the subject even before the results of the theory are laid down more comprehensively in a monograph which is in the early stages of planning. The attached bibliography was compiled in collaboration with J. D. Lawson and various participants in a workshop at Tulane University from April 2, through April 5, 1977.

Bibliography

The index in the first column refers to a rough classification into the following subdisciplines:

APP = Applications
CL = Continuous lattices
IT = Intrinsic topologies on semilattices and lattices
POTS = Partially ordered topological spaces
TSL = Topological semilattices
TL = Topological lattices

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