CONTINUOUS LATTICES, TOPOLOGY AND TOPOLOGICAL ALGEBRA

by

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In 1974 Jean Dieudonné wrote an article in la Gazette des Mathématiciens in which he philosophized on what he believed to constitute noble mathematics and servile mathematics. He arranged all the mathematics he could think of in a vertical hierarchy. General topology, located on the lowest level of the hierarchy appears to be a rather ignoble commodity. And where is lattice theory? Must we conclude from its absence that, having been banned from the hierarchy, it is totally without redeeming value?

Nevertheless, this lecture will be on topology and lattice theory. We will discuss under the name of continuous lattices a relatively novel class of lattices which emerged quite independently in various guises in different branches of mathematics ranging from computer science to category theory under the dictate of need. Recent research demonstrated its significance and elucidated its role as a link between different areas. For the purpose of this lecture we will restrict our attention primarily to bonds with topology.

Even Dieudonné acknowledged that any line of mathematics may escalate to a higher level of existence in his mathematical universe under the influence and imagination of a

new leader. Our story will be about the leadership of mathe-
maticians like J. D. Lawson and Dana Scott.

Introduction

    In order to define a continuous lattice, we observe
that in any partially ordered set we may define an auxiliary
transitive relation by x << y whenever for any subset A with
y $\leq$ sup A we find a finite subset F $\subseteq$ A with x $\leq$ sup F. The
relation x << y implies x $\leq$ y, but it may never hold for all
we know; if there is a smallest element 0, then 0 << x for
all x. More generally, for a finite partially ordered set
we have x << y iff x $\leq$ y. Otherwise, even in cases where
this relation is prevalent, it is not generally reflexive.
An element x with x << x is called *compact*.

    Now we can introduce the following definition

    Definition. A partially ordered set L is called a
continuous lattice iff
    (0) L is a complete lattice (i.e. every subset has a
        least upper bound)
    (1) ($\forall$x $\in$ L) x = sup{u $\in$ L: u << x}.
It is called an *algebraic lattice* iff it satisfies (1) and
    (2) ($\forall$x $\in$ L) x = sup{k $\in$ L:k << k $\leq$ x}.
The subset K(L) of compact elements is always a sup-semi-
lattice.

    A lattice is *arithmetic*, iff it is algebraic and the
set K(L) is a sublattice.

    The implications finite $\Rightarrow$ arithmetic $\Rightarrow$ algebraic $\Rightarrow$
continuous $\Rightarrow$ complete are clear, and none of these implica-
tions can be reversed. We will see much of most of these

in the sequel; each class is closed under forming arbitrary
products. Each complete chain is a continuous lattice.
The lattice of open sets of a locally compact space is con-
tinuous, where U << V if we have a compact set with $U \subseteq C \subseteq V$.
The lattice of open sets of a Boolean space is algebraic
with the compact open sets being precisely the compact
elements in the lattice. Scott's universe $P\omega = \{x | x \subseteq \omega\}$
for recursive function theory and the λ-calculus is an alge-
braic lattice. The lattices of all ideals of a ring, or of
subgroups of a group, or of congruences of a general algebra
are algebraic, the compact elements being the finitely gener-
ated objects. (So much for terminology.)

The lattice of two sided *closed* ideals of a C*-algebra
is continuous. The lattice of lower semicontinuous extended
functions f: X → $\mathbf{R} \cup \{\pm\infty\}$ on a compact space is continuous.
The concepts of algebraic and arithmetic lattices are tradi-
tional; the concept of a continuous lattice has been intro-
duced by D. Scott about 5 years ago and he baptized them.
They should not be mixed up with the continuous geometries
of von Neumann. The definition given above is about two
years old. Let me draw your attention to one feature which
sets these concepts apart from the cherished features of
classical lattice theory: They are asymmetrical insofar as
conditions (1) and (2) are not symmetric (i.e. invariant
under passage to the new relation $x \leq^{op} y$ iff $y \leq x$. In
fact $<<^{op}$ is not derived from $\leq^{op}$ as is << from $\leq$. We will
return to this motive of asymmetry time and again; it may be
one of the reasons that the fruitful exploitation of these
concepts is recent.

Some General Thoughts on the Relation Between

Topology and General Topology

Traditionally, topology and lattice theory touch in
several contexts. We list a few:

1) To each topological space X one associates the
lattice O(X) of all open sets (sometimes the sublattice
OC(X) of open closed subsets). There are ways to associate
with a lattice L a topological space called its spectrum
Spec L. One finds natural functions X → Spec O(X) and
L → O(Spec X) which may be interpreted and utilized in dif-
ferent ways: Firstly, X → Spec O(X) will reveal how much
information on the space X can be retrieved from knowing the
lattice O(X) and to what extent "topology without points"
is possible. The history of this line of investigation bears
the marks of Dowker, Papert and Papert, Bénabou, Isbell.
Secondly, L → O(Spec X) affords a method to represent lattices
as lattices of open sets. One would not expect a universally
applicable representation theory since O(X) is always a
distributive lattice whereas many important lattices (such
as the lattice of subgroups of a group, or the lattice of
projections of a Hilbert space (playing an important role
in the foundation of quantum mechanics)) are not distributive.
In many respects distributivity is to the lattice theoretician
what commutativity is to the group theoretician: There are
many interesting problems attached to the special class,
but others are entirely outside its scope. The usefulness
of the representation theory of lattices through this method
has the most venerable tradition in lattice theory, as is

illustrated by the classical Boole-Stone representation
theorem for Boolean lattices, which was expanded by Stone to
a wide class of distributive lattices.

2) On each space we have a quasi order given by $x \leq y$
iff $y \in \{x\}^-$ (iff for all open sets U one has $x \in U$ if
$y \in U$). This relation is a partial order iff the space is
T_0. Conversely, if a partial ordered set is given, there
are numerous ways of making a T_0-topology from the order.
(Example: Basic open sets: $\{x \mid \alpha \nleq x\}$, $\alpha \in L$).

Why would T_0-spaces deserve particular attention when
so many articles or lectures begin "we will assume through-
out that all spaces are Hausdorff" and when, for instance in
the theory of topological groups, T_0-separation already im-
plies Hausdorff complete regularity?

One principal motivation for considering T_0-spaces is
the spectral theory of commutative rings (algebraic geometry)
and of algebras such as Banach algebras, operator algebras
(functional analysis), or the study of congruences on algebras
of a general type (universal algebra). Specifically, let R
be a commutative ring, and let Spec R be the set of all prime
ideals P (i.e. R\P is a multiplicative semigroup), and call
a subset $h(X) = \{P \in \text{Spec } R: X \subseteq P\}$ the hull of $X \subseteq R$.
Trivally, $h(\cup_j X_j) = \cap_j h(X_j)$; but the primeness of P guarantees
also that $P \in h(x) \cup h(y)$ iff $x \in P$ or $y \in P$ iff $xy \in P$ iff
$P \in h(xy)$. The sets $\sigma(x) = \text{Spec } R \backslash h(x)$ thus form a basis of
open sets of a topology, called the *hull-kernel topology*.
It is at the heart of all spectral theory. The hull kernel
topology is T_0, but rarely better. Notice that the sets
$\sigma(x)$ are quasicompact and open.

If A is a C*-algebra, then the set $\hat{A}$ of unitary equiva-
lence classes of irreducible *-representations on a Hilbert
space is a non-separated topological space with the property
that the lattice Id A of closed two sided ideals of A is
isomorphic to $O(\hat{A})$.

If A is a universal algebra, belonging to an equational
class, then the lattice of congruences is a central object
of study in universal algebra. One may again assign to the
algebra a space, namely, the space of irreducible congruences.
Its topology is a hull-kernel topology. Again most spaces
arising in this way are not separated. Another reason to
consider T_o-spaces comes from classical considerations of real
valued functions. The study of lower semicontinuous func-
tions $f: X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is subsumed in continuous function
space theory if we admit on the range the T_o-topology of all
open intervals which are unbounded above.

3) A third motive relating topology and lattices arises
from the endeavor to introduce on a lattice in a natural way
topologies relative to which the operations $(x,y) \longmapsto xy =$
$x \wedge y$, $x \vee y$ are continuous. Much has been written on this
branch of topological algebra. Any totally ordered set can
serve as motivating example with the interval topology.

1. T_0 -Spaces and Continuous Lattices

Let us introduce some notation: For a poset L and a
subset $A \subseteq L$ we write $\uparrow A = \{x \in L: (\exists a)\ a \in A$ and $a \leq x\}$.
We write $\uparrow a$ in place of $\uparrow\{a\}$. The notation $\downarrow A$ is defined
dually. We write $\Downarrow x = \{y \in L | y \ll x\}$ and $\Uparrow x = \{y \in L | x \ll y\}$.

1.1. *Definition*. Let L be a poset. Then $U \subseteq L$ is

Scott-open, iff $\uparrow U = U$ and $(\forall D)$ D up-directed with sup $D \subseteq U$

$\Rightarrow$ $D \cap U \neq \emptyset$. The collection of all Scott-open subsets is

the *Scott-topology*. Every $L \backslash \uparrow x$ is Scott-open (whence the

Scott topology is T_o). The topology generated by all $L \backslash \uparrow x$,

$x \in L$ is called the *spectral topology*. It is likewise T_o.

We will see how it relates to hull-kernel topologies. The

least upper bound of the two topologies will be called the

Lawson topology.

Notice that the spectral topology is not the opposite

of the Scott topology. Thus the Lawson topology is not a

symmetric concept. Every Scott-open set U is Lawson-open, and

by 1.1, the converse is true if $U = \uparrow U$ holds; thus a set

U with $\uparrow U = U$ in a complete lattice is Scott open iff it is

Lawson open. We note that $y \in$ int $\uparrow x$ (with int denoting

interior relative to the Scott- or the Lawson-topology)

implies $x \ll y$. The converse holds in continuous lattices.

We now have the following theorem:

1.2. *Theorem* (1977). *Let L be a complete lattice.*

Then the Lawson-topology on L is quasicompact T_1 *and the*

second of the following statements implies the first

(1) *The Lawson-topology is Hausdorff.*

(2) *L is a continuous lattice.*

Proof. The subbasic sets of the Lawson topology are of

the form U with a Scott open U and $L \backslash \uparrow x$, $x \in L$. Let

$L \subseteq \cup_i U_i \cup \cup_j (L \backslash \uparrow x_j) = \cup_i U_i \cup (L \backslash \cap_j \uparrow x_j)$. Suppose no finite

subfamily covers. For any finite set J of indices j, let

$x_J = \sup_J x_j$. Then for all i and all J there is an s with

$s \notin U_i$ and $s \in \uparrow x_J$. Thus $x_J \notin U_i$. If $x = \sup x_J = \sup x_j$, then $x \notin L\backslash\uparrow \sup x_j = U_j(L\backslash\uparrow x_j)$. Hence there is an i with $x \in U_i$. By the definition of the Scott topology, there is a J with $x_J \in U_j$, and this is a contradiction. Thus the Lawson topology is quasicompact. If we have $x \not\leq y$, then $L\backslash\downarrow x$ is a neighborhood of y not containing x and $L\backslash\uparrow y$ is a neighborhood of x not containing y. Thus the Lawson-topology is T_1.

   (2) $\Rightarrow$ (1): Let $x \not\leq y$; by (2) find $u << y$ and $x \not\leq u$. Then $L\backslash\uparrow u$ and $\uparrow u$ are disjoint neighborhoods of x and y, respectively.

Remark. If L is meet continuous (s.1.6 below), then (1) <=> (2). Indeed if $x < y$, then by (1) there is a basic neighborhood $U\backslash\uparrow F$ of x, U Scott-open and F finite, such that y is in the interior of its complement, i.e. $y \in$ int $\uparrow F$. By meet continuity, int $\uparrow F = U\{\uparrow u : u \in F\}$. Hence $u << y$ for some $u \in F$, and $u \not\leq x$. The case $y \not\leq x$ is reduced to this one. Hence $y = \sup \downarrow y$.

   This is a first characterization of continuous lattices among complete lattices by topological means. The first part of the proof is due to D. Scott.

   We noticed that on any posets there exist T_0-topologies. Let us now start from T_0-spaces and detect under what conditions one returns to lattices.

   All spaces are T_0 from here on out. We present Scott's approach (1972).

   1.3. *Definition.* A space Z is called *injective* iff for every pair $X \subseteq Y$ of spaces every continuous function $f: X \to Z$ extends to a continuous function $F: Y \to Z$.

This is equivalent to saying that Z is a retract of every containing space.

Note. This definition is categorical in the following sense: Let A be a category and J a class of monics. Then Z $\in$ ob A is a *J-injective* iff for every pair of morphisms j: X $\rightarrow$ Y and f: X $\rightarrow$ Z with j $\in$ J there is a morphism F: Y $\rightarrow$ Z with f = Fj. If $A = T_o$ is the category of T_o-spaces and J the class of embeddings, then the injectives of 1.3 are precisely the J-injectives.

Examples. Trivially every singleton space is injective. The lattice 2 = {0,1} with the Scott topology has {1} as the only non-trivial open set (the "Sierpinsky space"); it is easily seen to be injective. Indeed a continuous function f: X $\rightarrow$ 2 is just the characteristic function of an open set and thus, by the definition of induced topology, extends.

Products of injectives are injectives, as are retracts. Since every T_o-space X is embeddable into $2^{\text{Top}(X,2)}$, every space is embeddable into an injective one and the retracts of the spaces 2^M are precisely the injective ones.

Here then is Scott's theorem:

1.4. *Theorem* (Scott 1972). *Let L be a T_o-space with the partial relation $\leq$ defined by "x $\leq$ y iff for all open sets U we have x $\in$ U implies y $\in$ U."*

Then the following two conditions are equivalent:

(1) L is an injective T_o-space.

(2) L is a continuous lattice (relative to $\leq$).

Moreover, if these conditions are satisfied, then the topology on L is the Scott-topology associated with (L,$\leq$).

In order to speak of functions in greater detail, we
denote the full subcategory of the category <u>Top</u> of topologi-
cal spaces and *continuous* maps whose objects are injective
T_0-spaces (hence *continuous* lattices) by <u>Cont</u>. One of Scott's
main discoveries was that <u>Cont</u> is a cartesian closed category.
This does not only allow the creation of new continuous lat-
tices by forming products, but also by forming function spaces
which was one of the main motivations for Scott in his
creating models for the λ-calculus. (We will speak on the
formation of quotients and subobjects later when we have a
more subtle understanding of special types of morphisms.)

1.5. *Lemma.   A function* f: L → L' *between two complete
lattices is Scott-continuous iff it preserves sups of up-
directed sets. A function* f: L_1 × L_2 → L *of several varia-
bles (with continuous lattices* L_1, L_2 *and L) is continuous
with the product of the Scott-topologies iff it is continu-   y
ous in each variable separately.* (D. Scott).

The result about separate and joint continuity of multi-
variable functions is somewhat delicate. It seems unknown
whether it is true for complete lattices in general. After
a recent observation by Hofmann and Gierz it suffices that
L_1, L_2 are complete and the Scott topology $O(L_2)$ of L_2 is a
continuous lattice.

The sup operation (x,y) ⊢→ x ∨ y: L × L → L is Scott
continuous in any complete lattice. (This does not say that
it is continuous for the product topology on L × L which is
possibly coarser than the Scott topology on L × L. For both
topologies to be equal it suffices that O(L) be continuous.)

This fails for the inf operation in general. Hence one is
challenged to add a definition:

1.6. *Definition*. A lattice is called *meet continuous*
(MC) iff it is complete and $(x,y) \longmapsto xy: L \times L \to L$ is
continuous. This latter condition is equivalent to the
following condition:
(MC) For each chain $D \subseteq L$ and each $x \in L$ we have x sup D =
sup xD.

Recall that a complete lattice is *Brouwerian* iff it
satisfies (BC): For each subset $D \subseteq L$ and each $x \in L$ we have
x sup D = sup xD. Thus every Brouwerian complete lattice is
MC, and a meet continuous lattice is Brouwerian iff it is
distributive. In particular, all lattices O(X) are Brouwerian,
hence MC.

From our later observations it will be immediate (and
Scott showed a direct proof) that every continuous lattice
is meet continuous; the converse is not true. Thus we have
refined the hierarchy of lattices which we gave in the begin-
ning. We also observe as a consequence of 1.2 and 1.6

1.7. *Proposition*. *If L is a meet continuous lattice,
then L is a quasicompact T_1-topological semilattice (relative
to inf and the Lawson-topology), provided O(L) is continuous.*

Proof. Let m: $L \times L \to L$ be the multiplication
$(x,y) \longmapsto xy$. If $U \subseteq L$ is Scott open, then $m^{-1}(U)$ is open
since L is meet continuous. If $x \in L$ is arbitrary, then
$m^{-1}(L\backslash\uparrow x) = L \times L\backslash\{(s,t) \in L \times L | st \geq x\} = L \times L\backslash\uparrow(x,x)$ is
spectral-open. (O(L) continuous $\Rightarrow$ O(L $\times$ L) = O(L) $\times$ O(L).)

From 1.7 and 1.2 we draw the corollary that a continuous lattice is a compact topological semilattice in the Lawson-topology.

If X is a topological space, then O(X) is meet continuous. If O(X) is a continuous lattice, then O(X) is a compact T_2 topological ∩-semilattice, and it is a topological lattice for the Scott topology. As Isbell would say "any continuous topology is a topological topology," which makes more sense than "a rose is a rose is a rose." He points out, too, that contrary to what had been previously asserted by Scott, himself and myself, a topology need not be topological in general.

If X is an injective T_o-space, then the lattice O(X) of Scott open even subsets is isomorphic to Cont(X,2) (where 2 has the Scott topology). In the process of establishing that Cont is cartesian closed, Scott showed:

1.8. *Lemma. If L,L' ∈* Cont, *then the topology of pointwise convergence is the Scott-topology for the pointwise partial order (f ≤ g iff f(x) ≤ g(x)) and* Cont(L,L') ∈ Cont *(i.e.* Cont(L,L') *is a continuous lattice).*

In particular, then if X ∈ Cont, then O(X) ≅ Cont(L,2) is a continuous lattice. We notice:

1.9. *Lemma. For a topological space* X *the following statements are equivalent:*

   (1) O(X) *is a continuous lattice.*

   (2) For each point x *and each open neighborhood* U *of* x *there is an open neighborhood* V ⊆ U *of* x *such that*

every filter on V *clusters in* U *(i.e. every ultra-*
filter on V *converges in* U*).*

Isbell calls these spaces *semi-locally bounded,* while
A. S. Ward called them *quasi locally compact.* For the present
purposes I will adhere to the following less colorful defini-
tion:

1.10. *Definition.* A space will be called a CL-*space*
iff O(X) is a c̲ontinuous l̲attice.

We have seen that all injective T_0-spaces are CL-spaces.
From 1.6 it is clear (and B. J. Day and G. M. Kelly are on
record (1970) for this observation in a different terminology)
that every locally quasicompact space is a CL-space (where
a space is *locally quasicompact* if every point has a basis
of quasicompact neighborhoods). We will say more on this
later. But in the context of function spaces we are now
ready for Isbell's theorem which is the conclusive word on
function spaces and continuous lattices:

1.11. *Theorem.* (Isbell 1975) *Let* L *be a complete*
lattice and X *a space, and consider* L *with its Scott topology*
(making it into a T_0-*space). Consider* S = Top̲(X,L) *with the*
pointwise partial order. Then the following statements are
equivalent:

  (1) X *is a* CL-*space, and* L *a continuous lattice.*

  (2) S *is a continuous lattice (i.e. an object of* Cont̲ *in*
      its Scott topology which is the topology of point-
      wise convergence).

Proof.   If (1) is satisfied, then $\underline{\text{Top}}(X,2) \cong O(X)$ is

a continuous lattice. Now L is a retract of some 2^M, but

$\underline{\text{Top}}(X,-)$ preserves products and retracts. Thus $\underline{\text{Top}}(X,L)$ is

a retract of an injective space, hence is injective. So

(2) holds. Conversely, suppose (2). Consider the retraction

f $\longmapsto$ sup f: $\underline{\text{Top}}(X,L) \to L$ which is right adjoint to const:

$L \to \underline{\text{Top}}(X,L)$, const(s)(x) = s: Indeed s $\geq$ sup f iff

const s $\geq$ f. But const preserves sups, hence is continuous.

Thus L is a retract of $\underline{\text{Top}}(X,L)$ hence is a continuous lattice.

Now 2 is a retract in $\underline{\text{Cont}}$ of L, hence $\underline{\text{Top}}(X,2)$ is a retract

in $\underline{\text{Top}}$ of $\underline{\text{Top}}(X,L)$, hence is injective and so X is a CL-space.

We notice that the augmented reals $\widetilde{\mathbf{R}} = \widetilde{\mathbf{R}} \cup \{\pm\infty\}$ form

a continuous lattice and that for any locally compact space

X, the space $\underline{\text{Top}}(X,(\widetilde{\mathbf{R}}, \text{Scott}))$ is exactly the space $LC(X, \widetilde{\mathbf{R}})$

of extended lower semicontinuous functions. Isbell's Theorem

(1.11) together with 1.2 then says that there is a compact

function space topology on $LC(X, \widetilde{\mathbf{R}})$ relative to which inf

is a continuous operation. I do not know whether this fact

is known in classical analysis.

We mentioned the observation by Day and Kelly that

locally quasicompact spaces X have a continuous lattice of

open sets. A complete characterization of CL-spaces appears

to be difficult after Isbell and Hofmann and Lawson found

CL-spaces in which every quasicompact set has empty interior.

Nevertheless something can be said. Let us recall that a

space is primal (or sober) if every closed set which is

irreducible (i.e. cannot be the union of two proper subsets)

has a dense point. Every T_o-space allows a universal

embedding X → X̌ into a sober space. We now have the follow-
ing theorem.

   1.12. *Theorem* (Hofmann and Lawson, 1977). *For a*
T_o-*space* X *the following statements are equivalent:*

   (1) X *is a* CL-*space.*

   (2) The sobrification of X *is locally quasicompact.*

   Let L be a complete lattice and let Spec L be the set
of all primes p < 1 with the topology induced by the spectral
topology; this topology is also called the hull-kernel to-
pology. For instance, X̌ = Spec O(X), and the embedding of X
is given by x ⊢⟶ X\{x}. Let us call a subset X of a lat-
tice L *order generating* if every element is the inf of a
subset of X. We then have

   1.13. *Theorem* (Hofmann and Lawson, 1977). *Let* L *be a
distributive continuous lattice. Then* Spec L *is order gen-
erating and the map* x ⊢⟶ Spec L\↑x: L → O(Spec L) *is an
isomorphism of lattices. Further,* Spec L *is a sober locally
quasicompact space, and all such spaces are so obtained.*
   More specifically, any order generating subspace X *of*
Spec L *is a* CL-*space, and every* CL-*space is so obtained.*

2. Compact Semigroups
   Workers in topological algebra and functional analysis
are interested in compact topological and semitopological
semigroups. Let us accept for the moment that the abelian
ones play a particularly important role. Suppose that S is
a monoid (= semigroup with identity). Then E(S) =
$\{s \in S | s^2 = s\}$ is a submonoid and if S is topological then

E(S) is closed (if S is semitopological, this is not generally
the case). In the structure theory of S the monoid E(S) is
an important functorial invariant. Let us mention an example
even though it does not totally fit the topological case: If
G is a locally compact abelian group and M(G) its measure
algebra, then its structure space ΔM(G) according to J. Taylor
is a compact semitopological abelian semigroup in which cer-
tain elements of E(ΔM(G)) (so called critical idempotents)
correspond bijectively to locally compact group topologies on
G refining the given one and they form the index set for an
explicit sum representation of the cohomology of ΔM(G). In
fact ΔM(G) is the character semigroup of a compact abelian
topological semigroup.

    2.1. *Definition.* A *compact topological semilattice*
is a compact space with a continuous commutative idempotent
multiplication with identity.

    Apart from the applications, compact topological semi-
lattices are, in the simplest cases, easy to come by and
therefore yield a useful supply of examples in the theory
of compact semigroups. We will always use the word compact
semilattice for compact commutative idempotent monoid. Every
product of compact semilattices is one, every closed sub-
semilattice of a semilattice is again one. Quotients of
compact semilattices are compact semilattices, so they form
a quasivariety. All totally ordered order complete sets
(such as the unit interval I and the Cantor set C) are compact
semilattices under min. By the preceding they co-generate a
large subvariety if not (as would be a priori conceivable

and was indeed believed for some years) the entire class.
It was observed early that compact semilattices exhibit some
interesting phenomena which are not present with compact
groups: The Cantor semilattice C allows I as quotient (via
Cantor's gap closing function) but dim C = 0 and dim I = 1
(by anybody's dimension count), and so there are dimension
raising morphisms around. This was known to Wallace and
Koch around 1960 but was only recently clarified completely
(Hofmann, Mislove, Stralka 1973). Studies in compact
topological lattices and semilattices were undertaken by
Wallace's school since the fifties.

   We mention one basic theorem due to R. J. Koch. Let us
write CS for the category of compact semilattices with con-
tinuous semilattice morphisms preserving identities.

   2.2. *Theorem. Let S ∈* CS. *Then the following state-
ments are equivalent:*

   (1) S is connected.

   (2) If k ∈ S *is a local minimum (i.e.* k *is isolated in*
      Sk) *then* k = 0.

   (3) For each s ∈ S *there is a connected compact chain*
      T ⊆ Ss *with* {s,0} ⊆ T.

   (4) S is acyclic (over any coefficient module).

The topological algebra of semilattices began moving
strongly when J. D. Lawson arrived and attacked the question
whether or not compact semilattices had a character theory
somewhat like that of compact abelian groups, where morphisms
into the circle group were replaced by morphisms into the
semilattice I. The first problem which poses itself in this

context is whether or not the morphisms S → I separate the
points of a compact semilattice. This is tantamount to ask-
ing whether any compact semilattice is a subsemilattice of
some "cube." Lawson proved the following theorem:

   2.3.   *Theorem.   Let* S ∈ CS, *the following statements
are equivalent:*

   (1) CS(S, I) *separates the points of* S.

   (2) [resp. (2')] S *has small [open] semilattices (i.e.
      every point has a basis of [open] neighborhoods* U
      with $U^2 \subseteq U$).

   (3) S *is ultra-uniform, i.e. the uniform structure of*
      S *has a basis of neighborhoods* W *of the diagonal
      in* S × S *which is a subsemilattice of* S × S.

   It was believed for some time that all CS-objects
satisfied these conditions. But Lawson established the
existence of a topologically one dimensional semilattice on
which all morphisms into the unit interval are constant.
That was in 1970, and it is hard even today to come by
examples of CS-objects which do not satisfy the conditions
of Theorem 2.2. The recent discovery by J. W. Roberts of
compact convex subsets without extreme points in topological
vector spaces yields a class of such examples, since for every
compact convex subset X of a topological vector space the set
S of all compact convex subsets A ⊆ X is a compact semilattice
under the operation (A,B) ⊢⟶ AB = closed convex hull of
A ∪ B (R. A. Jamison, J. D. Lawson). The semilattice satisfy-
ing the conditions of 2.2 are called *compact Lawson semi-
lattices.* Let us denote the full subcategory of CS consisting

of compact Lawson semilattices by $\underline{CL}$. By Lawson's discovery
there is a gap between $\underline{CS}$ and $\underline{CL}$. One wonders about the
category of all compact semitopological semilattices (in
which multiplication if only separately continuous). But
here we have a more recent result of Lawson's:

    2.4. *Theorem* (1976). *Any compact semitopological semi-*
lattice is topological.

    The following question poses itself naturally:

    Let $S \in \underline{CS}$; what are topological (or algebraic) hypothe-
ses on S which entail $S \in \underline{CL}$? Lawson showed that dim $S < \infty$
and local connectivity are enough. In fact he proves more.
We need a few ideas on the connectivity relation on spaces.
Let C_x be the relation of connectivity on the space X, i.e.
$x \ C_x \ y$ iff there is a connected subspace $Y \subseteq X$ with $x,y \in Y$.
We say that a subspace $Y \subseteq X$ is *fitted into* X if $C_Y =$
$C_X | Y \ (= C_X \cap (Y \times Y))$. We say that an equivalence relation R
on a space is lower semicontinuous if $R(A)^- = R(R(A)^-)$ for
all $A \subseteq X$. Finally we say that a neighborhood W of x in X
is *well-fitted* iff (i) W is compact, (ii) each point $v \in$ int W
has a basis of compact neighborhoods V such that V is fitted
into W and (iii) C_W is lower semicontinuous.
    We notice that every space which is locally a product of
a compact zero dimensional space and a locally connected
space has a well-fitted neighborhood around each point.

    2.5. *Theorem* (Lawson 1977). *If* $S \in \underline{SC}$ *has a well-*
fitted finite dimensional neighborhood at each point, then
$S \in \underline{CL}$.

In particular, one retrieves the old fact, due to a

theorem of Numakura's:

2.6. *Remark.* *If* S $\in$ CS *is zero dimensional, then*

S $\in$ CL.

The full category of zero dimensional objects in CS is

called Z.

While character theory did not work too well for CS,

and not particularly well for CL, it works superbly for Z.

In fact we have:

2.7. *Theorem.* *The category* Z *is dual to the category*

of semilattices S *under the functors* S $\longmapsto$ CS(S,2): Z $\rightarrow$ S

and T $\longmapsto$ S(T,2): S $\rightarrow$ Z, *where the hom sets are given the*

pointwise structures.

This is a parallel to Pontryagin duality for compact

abelian groups; it is due to various authors such as Austin,

Schnepermann, Bowman and was investigated in detail by Hof-

mann, Mislove, Stralka.

What has all of this got to do with lattice theory? A

very primitive observation is that the partial order given

on a compact semilattice S by x $\leq$ y iff xy = x gives S the

structure of a complete lattice. In fact it is not hard to

see that we obtain a meet continuous lattice. If we denote

with MC the category of all meet continuous lattices with

morphisms preserving arbitrary infs and sups of updirected

sets (i.e. Lawson-continuous morphisms) then we have CS $\subseteq$ MC

(considering any obvious forgetful functor an inclusion).

The question now arises: Is every CS-object in fact a

continuous lattice?   The answer is no and the precise
circumstances are exactly known by a result first estab-
lished by Hofmann and Stralka (1974).

   2.8.   *Theorem.   Let* L *be a complete lattice.   Then the
following statements are equivalent:*

   (1) L *is a continuous lattice.*

   (2) There is on L *a compact Hausdorff topology such that*
       L *is a compact Lawson semilattice.*

*If these conditions are satisfied, then there is only one
such topology, the Lawson-topology.*

   We add to this theorem our earlier observations on
morphisms:

   Let f: L → L' *be a function between two lattices.   Then
the following statements are equivalent:*

   (I) f *is a morphism of compact Lawson semigroups (rela-
       tive to the Lawson-topologies).*

   (II) f *is a morphism of continuous lattices (as specified
       after 1.14).*

Notice that the appearance of a semilattice in (2) is evi-
dence once more of an asymmetric condition of work on the
lattice L.

   Since every continuous lattice has exactly one compact
semilattice structure (which indeed makes it into a compact
Lawson semilattice) and since the underlying lattice structure
of a compact Lawson semilattice is that of a continuous lat-
tice, the categories of continuous lattice and of compact

Lawson semilattices are literally equal (not only naturally

isomorphic). Our nomenclature is therefore justified: CL

reads equally well as "continuous lattices" and as "compact

Lawsons." We now have the following hierarchy of categories

of complete lattices: CL $\subseteq$ CS $\subseteq$ MC. As we saw, very little

has been known on CS\CL, and the same is true for MC\CS.

The most recent result in this direction emerges from

contributions by Gierz, Hofmann, Lawson, Mislove and reads

as follows:

2.9. *Theorem (1977). Let L be a meet continuous lat-*

tice. Then the following are equivalent:

(1) *L carries a (unique) compact semilattice topology*

   (i.e. L $\in$ CS)

(2) *O(L) (the lattice of Scott-open sets) is a continu-*

   ous lattice (i.e. O(L) $\in$ CL)

(3) *In the Scott topology, L is a locally quasicompact*

   space.

Furthermore, the following are equivalent:

(I) *L carries a (unique) compact Lawson semilattice*

   topology (i.e. L $\in$ CL)

(II) *O(L) $\in$ CL and every element of O(L) is a sup of*

   coprimes

(III) *In the Scott topology, L is a locally quasicompact*

   space with a basis of open filters.

We still know little on sufficient conditions which would

allow us to introduce on an MC lattice a CS or a CL topology.

The only thing we have so far is a result of Lawson's

(1976):

   2.10.   *Theorem.   If* S ∈ MC, *and if* S *does not contain
a copy of a free semilattice in infinitely many generators,
then* S ∈ CL.

   This is just about all we know about what happens above
CL.   And below?   Here the situation is quite good.   In fact
we have the following result due to Hofmann, Mislove and
Stralka, which historically preceded Theorem 2.8:

   2.11.   *Theorem.   Let* L *be a complete lattice.   Then the
following statements are equivalent:*
   (1) L *is an algebraic lattice.*
   (2) There is on L *a compact zero dimensional Hausdorff
        topology such that* L *is a compact semilattice.*

   Thus the full subcategory of algebraic lattices in CL
is precisely Z, the category of compact zero dimensional
semilattices.   More information can be given on the Pontryagin
duality of Z and S first mentioned in Theorem 2.7.

   2.12.   *Proposition.   Let* L ∈ Z.   *Then the character
semilattice* $\hat{L}$ = CL(L,2) *of* L *is isomorphic to the sup-semi-
lattice* K(L) *of compact elements of* L *under the map*
χ $\longmapsto$ min χ^{-1}(1): $\hat{L}$ → K(L).   *An element* s ∈ L *is in* K(L)
*iff it is a local minimum in the Lawson-topology or the
Scott topology.*

   Let S be a semilattice written multiplicatively.   Then
the character semilattice $\hat{S}$ = S(S,2) is isomorphic to the
semilattice $\mathcal{J}$(S) of filters on S under the map χ $\longmapsto$ χ^{-1}(1):
$\hat{S}$ → $\mathcal{J}$(S).

There are functorial relations in the hierarchy of
lattices which point in the vertical direction. Thus it is
often useful to associate with a complete lattice L a $\underline{Z}$-ob-
ject PL which is simply the set of lattice ideals of L with
the containment as order. The compact elements are the
principal ideals and there is a canonical map r_L: PL $\rightarrow$ L
given by $r_L(J)$ = sup J. Since r_L is right adjoint to the
map x $\longmapsto$ $\downarrow$x, it preserves arbitrary sups. We have the
following theorem (Hofmann and Stralka 1974):

   2.13. *Proposition. Let* L *be a complete lattice and*
r_L: PL $\rightarrow$ L *the* SUP-map *given by* $r_L(J)$ = sup J. *Then*

   (a) r_L *is a lattice morphism iff* L $\in$ $\underline{MC}$.

   (b) r_L *preserves arbitrary infs (in addition to arbi-*
      trary sups) iff L $\in$ $\underline{CL}$.

   For a compact Lawson semilattice S the function
r_S: PS $\rightarrow$ S is a canonical way of representing S as a quotient
of a compact zero dimensional semilattice. Notice that the
proposition is yet another characterization of continuous
lattices. If L $\in$ $\underline{CL}$ then the right adjoint of r_L associates
with x $\in$ L the unique smallest dense ideal $\downarrow$x in $\downarrow$x.

Conclusion
   Only the relations between topology and topological
algebra were discussed in this lecture, and those only in a
fleeting way. We could not touch the emergence of continuous
lattices from the theory of computing which is in fact one
of their sources opened up by Dana Scott. In category
theory Alan Day and O. Wyler discovered that continuous

lattices are precisely the monadic algebras of filter and ultrafilter monads; this provides the link between continuous lattice theory and universal algebra. In recent months numerous occurrences of continuous lattices were discovered in functional analysis, and in fact R. Giles has noticed their appearance in his work in formalizing interpretive rules in the foundation of physics.

Research in the area is in full flux, and it would be too premature to propose a summary at this time. It may, however, be of use to present a bibliography of the subject even before the results of the theory are laid down more comprehensively in a monograph which is in the early stages of planning. The attached bibliography was compiled in collaboration with J. D. Lawson and various participants in a workshop at Tulane University from April 2, through April 5, 1977.

Bibliography

The index in the first column refers to a rough classi-fication into the following subdisciplines:

    APP = Applications
    CL   = Continuous lattices
    IT   = Intrinsic topologies on semilattices and lattices
    POTS = Partially ordered topological spaces
    TSL = Topological semilattices
    TL   = Topological lattices

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