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A LINEARIZATION OF SEMIFLOWS IN THE HILBERT SPACE l_2

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Abstract

Let $(\mathbb{R}^+, X, \alpha)$ be a semiflow on a compact metric space X satisfying: (1) The transition function $\alpha(t, \cdot): X \rightarrow X$ is one-to-one for every $t \in [0, \infty)$; (2) The semiflow shrinks the space X to a point, i.e., $\bigcap \{\alpha(t, X) : t \geq 0\}$ is a singleton. It is shown that, given any constant $c \in (0, 1)$, the semiflow $(\mathbb{R}^+, X, \alpha)$ can be embedded in the natural flow on the Hilbert space l_2 with similarity constant c , i.e., there exists a topological embedding $i: X \rightarrow l_2$ such that $i \circ \alpha(t, x) = c^t i x$ for every $x \in X$ and every $t \in \mathbb{R}^+$.

1. Introduction

By a semiflow $(\mathbb{R}^+, X, \alpha)$ we understand a continuous semigroup action $\alpha: \mathbb{R}^+ \times X \rightarrow X$ of the additive semigroup of non-negative reals $\mathbb{R}^+ = [0, \infty)$ on a nonempty topological space X . We say that a semiflow $(\mathbb{R}^+, X, \alpha)$ can be embedded in a flow (\mathbb{R}, Y, β) on a topological space Y if there exists a topological embedding $i: X \rightarrow Y$ such that the group action $\beta: \mathbb{R} \times Y \rightarrow Y$ restricted to $\mathbb{R}^+ \times i(X)$ coincides with the action of α , i.e., $i \alpha(t, i^{-1}y) = \beta(t, y)$ for every $t \geq 0$ and $y \in i(X)$. In the same sense we understand the phrase "embedding" of one semiflow into another or of one flow into another. Given $c > 0$ we denote by $(\mathbb{R}, l_2, \alpha_c)$ the natural flow on the Hilbert space l_2 with the similarity constant c , i.e., for every $t \in \mathbb{R}$

and every $x \in l_2$ we set $\alpha_c(t, x) = c^t x$. By the Hilbert cube H we understand the subset of l_2 defined by $\{x: x \in l_2, x = (x_1, x_2, \dots, x_n, \dots) \text{ with } |x_n| \leq 1/n \text{ for } n \geq 1\}$. We observe that if $c \in (0, 1]$ the Hilbert cube H is positively invariant which implies that $(\mathbb{R}^+, H, \alpha_c)$ is a semiflow on H .

In the sequel we shall also consider discrete semiflows determined by a space X and a continuous selfmap $f: X \rightarrow X$. To every semiflow $(\mathbb{R}^+, X, \alpha)$ there corresponds the discrete semiflow on X determined by $f: X \rightarrow X$ defined by $f(x) = \alpha(1, x)$ for every $x \in X$.

The objective of this note is to establish the continuous analog of the following result concerning discrete semiflows which is due to M. Edelstein [5] and the author [7].

Theorem A. Let $f: X \rightarrow X$ be a continuous map of a compact metric space X into itself satisfying:

- (i) f is one-to-one;
- (ii) f shrinks the space X to a point, i.e., $\bigcap \{f^n(X) : n \geq 1\}$ is a singleton.

Then given a constant $c \in (0, 1)$ there is a topological embedding $i: X \rightarrow l_2$ such that $i f(x) = c i(x)$ for every $x \in X$.

Motivated by this result we first define a class of semiflows.

Definition 1.1. A semiflow $(\mathbb{R}^+, X, \alpha)$ on a compact metric space X is said to be of class S iff

- (i) The transition function $\alpha(t, \cdot): X \rightarrow X$ is one-to-one for every $t \geq 0$;
- (ii) The action α shrinks the space X to a point, i.e., $\bigcap \{\alpha(t, X) : t \geq 0\}$ is a singleton.

Our theorem now reads

Theorem 1.1. Let $(\mathbb{R}^+, X, \alpha)$ be a semiflow of class S. Then, given $c \in (0, 1)$ there exists a topological embedding $i: X \rightarrow l_2$ such that

$$i\alpha(t, x) = c^t i(x) \text{ for } x \in X \text{ and } t \in \mathbb{R}^+.$$

We shall also show that the embedding $i: X \rightarrow l_2$ can be chosen in such a way that $i(X) \subset H$ so that the semiflow $(\mathbb{R}^+, X, \alpha)$ can be embedded in the semiflow $(\mathbb{R}^+, H, \alpha_c)$. This means that the semiflow $(\mathbb{R}^+, H, \alpha_c)$ is a universal semiflow in the sense of R. D. Anderson [1] and J. de Groot [6] for the family S.

The proof of this theorem will be given in the following three steps:

(1) Given a semiflow $(\mathbb{R}^+, X, \alpha)$ of class S we exhibit certain nice metric properties of $(\mathbb{R}^+, X, \alpha)$ relative to a suitable metric d on the underlying space X .

(2) We embed isometrically the semiflow $(\mathbb{R}^+, X, \alpha)$ in a flow (\mathbb{R}, Y, β) on a metric space (Y, d^*) .

(3) Considering trajectories $\gamma(y)$ in the flow (\mathbb{R}, Y, β) we construct geometrically, using the specific properties of the metric d^* on Y , a family $\{F_n: n \geq 1\}$ of solutions to the functional equation

$$F(\beta(t, x)) = c^t F(x).$$

The desired embedding $i: X \rightarrow l_2$ is then defined by setting $i(x) = (x_1, x_2, \dots, x_n, \dots)$ where $x_n = \frac{1}{n} F_n(x)$ for $x \in X$ and $n = 1, 2, \dots$.

2. Embedding of the Semiflow $(\mathbb{R}^+, X, \alpha)$ in a Flow

Definition 2.1. If X is a metrizable topological space we denote by $M(X)$ the set of all metrics on X inducing the topology of X . If (X, d) is a metric space then the statement $d^* \in M(X)$ means that d^* is topologically equivalent to d .

Definition 2.2. Let $(\mathbb{R}^+, X, \alpha)$ be a semiflow on a metric space (X, d) and let $c > 0$ be a positive constant. We say that $(\mathbb{R}^+, X, \alpha)$ is a similarity semiflow with the similarity constant c (or briefly c -semiflow) if for each $t \geq 0$ and every $x, y \in X$ we have

$$d(\alpha(t, x), \alpha(t, y)) = c^t d(x, y) \quad (*)$$

Requiring $(*)$ to be satisfied for every $t \in \mathbb{R}$ we define in the same way a similarity flow (\mathbb{R}, X, α) on a metric space (X, d) and call it a c -flow.

Lemma 2.1. Let $(\mathbb{R}^+, X, \alpha)$ be a semiflow on a metric space (X, d) satisfying

$$d(\alpha(n, x), \alpha(n, y)) = c^n d(x, y) \quad (1)$$

for every integer $n \geq 0$ and every $x, y \in X$.

Then there exists a metric $d^* \in M(X)$ relative to which $(\mathbb{R}^+, X, \alpha)$ is a c -semiflow.

Proof. Defining $d^*(x, y)$ by

$$d^*(x, y) = \sup\{c^{-s} d[\alpha(s, x), \alpha(s, y)]: s \geq 0\} \quad (2)$$

one observes easily that d^* is a metric on X . Due to equation (1) one observes that the expression $c^{-s} d[\alpha(s, x), \alpha(s, y)]$ regarded as a function of s is periodic with period 1. This implies that the definition of $d^*(x, y)$ given by (2) can be modified in two ways:

- (a) The supremum in (2) is taken over the set $s \geq s_0$
for arbitrary $s_0 \geq 0$.

(b) The supremum in (2) is taken over $[0,1]$.

From the form (b) we prove easily that $d^* \in M(X)$. Assuming this is not the case we would have a sequence $\{x_n\} \subset X$ and a point $x \in X$ with $d(x_n, x) \rightarrow 0$ and $d^*(x_n, x) \neq 0$. Approximating $d^*(x_n, x)$ by $d(\alpha(s_n, x_n), \alpha(s_n, x))$ where $s_n \in [0,1]$ for $n = 1, 2, \dots$ and passing if necessary to a subsequence we arrive readily to a contradiction.

To show that d^* satisfies the equation (*) we apply the definition of d^* to the points $\alpha(t, x)$ and $\alpha(t, y)$ where $t \geq 0$ is arbitrary. From (2) and the semigroup property we obtain $d^*[\alpha(t, x), \alpha(t, y)] = \sup\{c^{-s}d[\alpha(t+s, x), \alpha(t+s, y)]: s \geq 0\}$, which can be rewritten in the form

$$c^t \sup\{c^{-u}d[\alpha(u, x), \alpha(u, y)]: u \geq t\}.$$

But this expression equals $c^t d^*(x, y)$ as we see using the form (a) where we choose $s_0 = t$. This concludes our proof.

Lemma 2.2. Let the semiflow $(\mathbb{R}^+, X, \alpha)$ be of class S. Then for every $c \in (0, 1)$ there exists a metric $d^* \in M(X)$ relative to which $(\mathbb{R}^+, X, \alpha)$ is a c -semiflow.

Proof. Defining $f: X \rightarrow X$ by $f(x) = \alpha(1, x)$ for $x \in X$ we obtain a situation satisfying Theorem A, since

$$\cap\{f^n(X): n \geq 1\} = \cap\{\alpha(t, X): t \geq 0\}$$

due to the fact that $t_1 \leq t_2$ implies $\alpha(t_2, X) \subset \alpha(t_1, X)$. Thus, there exists a topological embedding $i: X \rightarrow l_2$ such that $i \circ f(x) = ci(x)$ for every $x \in X$. Denoting by $\|\cdot\|$ the norm in l_2 and defining the metric $d \in M(X)$ by $d(x, y) = \|i(x) - i(y)\|$ we see immediately that d satisfies equation (1) of Lemma 2.1. The conclusion follows from that lemma.

Definition 2.3. We say that a point $x \in X$ in a semiflow

$(\mathbb{R}^+, X, \alpha)$ is a start point (see [3]) if for every $y \in X$ and every $t > 0$ we have $\alpha(t, y) \neq x$, i.e., if the only semitrajectory $\gamma^+(y)$ to which x belongs is $\gamma^+(x)$. The set of start points of $(\mathbb{R}^+, X, \alpha)$ we denote by B and call it the basis of $(\mathbb{R}^+, X, \alpha)$.

Lemma 2.3. Let the semiflow $(\mathbb{R}^+, X, \alpha)$ be of class S . Then $(\mathbb{R}^+, X, \alpha)$ satisfies the following two properties:

- (1) $(\mathbb{R}^+, X, \alpha)$ has a unique point of equilibrium $a \in X$, i.e., $\alpha(t, a) = a$ for every $t \geq 0$.
- (2) The basis B of $(\mathbb{R}^+, X, \alpha)$ is nonempty (provided $X \neq \{a\}$) and for every $x \neq a$ there is a unique $b \in B$ for which $x = \alpha(t, b)$ for some $t \geq 0$.

Proof. The property (1) follows from the definition of the class S setting

$$\{a\} = \cap \{\alpha(t, X) : t \geq 0\}.$$

To prove (2) we represent the space X in the form of a disjoint union $X = \cup \{A_n : n \geq 0\} \cup \{a\}$ where $A_n = \alpha(n, X) - \alpha(n+1, X)$ for $n = 0, 1, \dots$. Assume now that $x \in X$ and $x \neq a$. Then $x \in A_n$ for some $n \geq 0$. Defining the set T of nonnegative reals by

$$T = \{t : \text{there is } y \in X \text{ with } x = \alpha(t, y)\},$$

we observe that T is nonempty ($0 \in T$) and that T is bounded by $n + 1$ since otherwise $x \in \alpha(n+1, X)$ which is not the case. Setting $t = \sup T$ and using compactness of X we obtain that $x = \alpha(t, b)$ for some b which must be a start point.

Theorem 2.4. Given $c \in (0, 1)$ and a semiflow $(\mathbb{R}^+, X, \alpha)$ of class S one can embed $(\mathbb{R}^+, X, \alpha)$ in a c -flow (\mathbb{R}, Y, β) on some metric space (Y, d^*) .

Proof. From Lemma 2.2 follows that there exists a metric $d \in M(X)$ relative to which (R^+, X, α) is a c-semiflow, i.e., the equation

$$d[\alpha(t, x_1), \alpha(t, x_2)] = c^t d(x_1, x_2) \quad (*)$$

is satisfied for all $x_1, x_2 \in X$ and all $t \geq 0$. On the other hand Lemma 2.3 says that the basis B of (R^+, X, α) is nonempty (except in the trivial case that $X = \{a\}$ in which case the statement is trivially true) and that every element $x \in X$ distinct from the equilibrium point a can be uniquely presented in the form $x = \alpha(t, b)$ with $b \in B$.

We define the set Y as $R \times B \cup \{a\}$ identifying the points $y \in Y$ of the form $y = (s, b)$ with $s \geq 0$ with the points $\alpha(s, b)$ of X . The group action $\beta: R \times Y \rightarrow Y$ on Y we define by setting

$$(1) \beta(t, a) = a \text{ for every } t \in R.$$

$$(2) \beta[t, (s, b)] = (s+t, b)$$

for every $t \in R$ and $(s, b) \in R \times B$. Now we extend the metric d on X to a metric d^* on Y as follows.

(1) We set

$$d^*[(s, b), a] = c^{-s} d(b, a)$$

for every $(s, b) \in R \times B$.

(2) Assume $(s_1, b_1), (s_2, b_2) \in R \times B$. If $s_1 \geq 0$ and $s_2 \geq 0$ we set $d^*[(s_1, b_1), (s_2, b_2)] = d[\alpha(s_1, b_1), \alpha(s_2, b_2)]$. In opposite case we set $d^*[(s_1, b_1), (s_2, b_2)] = c^s d[\alpha(s_1 - s, b_1), \alpha(s_2 - s, b_2)]$ where $s = \min\{s_1, s_2\}$.

It is a matter of straightforward verification that the validity of the equation (*) is extended to the metric space (Y, d^*) which had to be shown.

Remark 2.1. From the construction of the flow (R, Y, β)

does not follow whether the topology of its underlying space Y generated by the metric d^* depends on the choice of the metric $d \in M(X)$ or not. The question is still open but for our purposes irrelevant.

Remark 2.2. Due to the validity of (*) all the trajectories $\gamma(y)$ of the flow (R, Y, β) have as the only limit point the equilibrium point a . Thus, for every point $y \in Y$ the set $\gamma(y) \cup \{a\}$ is closed.

3. Proof of the Theorem

To obtain the desired embedding $i: X \rightarrow l_2$ we consider the functional equation

$$F(\beta(t, y)) = c^t F(y) \quad (1)$$

on the flow (R, Y, β) obtained by Theorem 2.4.

All we need is to construct a countable family $\{F_n: n \geq 1\}$ of continuous solutions to the equation (1) satisfying:

- (i) For each $n \geq 1$ and each $x \in X$ $F_n(x) \in [0, 1]$.
- (ii) The family $\{F_n\}$ is point-separating, i.e., given two distinct points $x_1, x_2 \in X$ there is some $n \geq 1$ for which $F_n(x_1) \neq F_n(x_2)$.

The desired embedding is then defined by setting $i(x) = (x_1, x_2, \dots, x_n, \dots)$ where $x_n = \frac{1}{n} F_n(x)$ for $n = 1, 2, \dots$.

In the sequel by (R, Y, β) we understand the flow obtained in Theorem 2.4 assuming the constant $c \in (0, 1)$ and the semiflow (R^+, X, α) are given.

Definition 3.1. If $\gamma(y)$ is the trajectory through a point $y \in Y$ in the flow (R, Y, β) we denote by $F(x, y)$ the

distance $d^*(x, \gamma(y)) = \inf\{d^*[x, \beta(t, y)]: t \in \mathbb{R}\}$ between a point $x \in Y$ and the trajectory $\gamma(y)$.

Lemma 3.1. For any point $y \in Y$ the function $F(x, y)$ is continuous in x and satisfies the equation (1), i.e., $F[\beta(t, x), y] = c^t F(x, y)$ for every $t \in \mathbb{R}$ and $x \in Y$.

Proof. Continuity of $F(x, y)$ follows directly from the definition. Since $\gamma(y) \cup \{a\}$ is a closed set, (see the Remark 2.2) we conclude that $F(x, y) = 0$ if and only if $x \in \gamma(y)$ or $x = a$. But in this case $F[\beta(t, x), y]$ is also zero and our equation is satisfied. Assume therefore that $x \notin \gamma(y) \cup \{a\}$. Since the flow (\mathbb{R}, Y, β) satisfies the similarity equation (*) relative to d^* and since the number $F(x, y)$ can be approximated by $d^*[x, \beta(s, y)]$ for a suitable $s \in \mathbb{R}$ with any degree of accuracy, we obtain from the definition of $F(x, y)$ the inequality $F[\beta(t, x), y] \leq c^t F(x, y)$ valid for every $t \in \mathbb{R}$.

Using the group property of the flow (this was the only reason why we needed the embedding of $(\mathbb{R}^+, X, \alpha)$ in a flow) we may interchange the role of x and $\beta(t, x)$ to obtain the opposite inequality which completes our proof.

Assume now without loss of generality that the underlying metric space (X, d) of the semiflow $(\mathbb{R}^+, X, \alpha)$ has the diameter ≤ 1 . Since X is compact the basis B of X is some separable subset of the metric space (Y, d^*) in which the space (X, d) is isometrically embedded. Suppose first that the set B is infinite and that $\{b_n\}$ is a dense sequence in B . We now consider the family of trajectories $\{\gamma(b_n): n \geq 1\}$ passing through the points $b_n, n \geq 1$ and the corresponding

family of functions $\{F_n : n \geq 1\}$ where the functions $F_n : Y \rightarrow \mathbb{R}$ are defined by $F_n(x) = F(x, b_n)$ for $x \in Y$ and $n = 1, 2, \dots$.

Due to the fact that the diameter of X is ≤ 1 we observe that the restriction of F_n to X maps X into $[0, 1]$ for $n = 1, 2, \dots$, which proves that the family $\{F_n : n \geq 1\}$ satisfies the condition (i). It is now easy consequence of density of $\{b_n\}$ to establish the point-separating property of the family $\{F_n : n \geq 1\}$. Given two distinct points $x_1, x_2 \in X$ two cases have to be considered:

(1) x_1 and x_2 are situated both on some trajectory $\gamma(b)$ through some point $b \in B$. But then obviously we have $F_n(x_1) \neq F_n(x_2)$ for every $n = 1, 2, \dots$.

(2) x_1 and x_2 are not on the same trajectory. Then Lemma 2.3 implies existence of distinct elements b_1^* and b_2^* in B such that $x_1 = \beta(t_1, b_1^*)$ and $x_2 = \beta(t_2, b_2^*)$ for some $t_1, t_2 \geq 0$. Since b_1^* and b_2^* are distinct one can select a subsequence $\{b_{k(n)}\}$ of the sequence $\{b_n\}$ converging to b_1^* . This implies that the numbers $F_{k(n)}(x_1)$ tend to zero whereas the numbers $F_{k(n)}(x_2)$ do not. Thus, there is a suitable index n for which $F_n(x_1) \neq F_n(x_2)$ which establishes the property (ii) of the family $\{F_n\}$.

The embedding $i: X \rightarrow H \subset l_2$ of the semiflow $(\mathbb{R}^+, X, \alpha)$ in the semiflow $(\mathbb{R}^+, H, \alpha_c)$ on the Hilbert cube H is now defined by $i(x) = (x_1, x_2, \dots, x_n, \dots)$ where $x_n = \frac{1}{n} F_n(x)$ for $n = 1, 2, \dots$. This concludes the proof of our theorem in case that the basis B of $(\mathbb{R}^+, X, \alpha)$ is infinite.

It remains to discuss the case when B is finite: If B consists of one element only, say b , then our method does not apply since $F(x, b) = 0$ for every $x \in X$. But then X

consists of one single semi-trajectory $\gamma^+(b)$ and can be identified with $[0,1]$ where 0 is the point a and 1 stands for b , and the action α is defined by $\alpha(t,x) = c^t x$ for $t \geq 0$ and $x \in [0,1]$. In the case when B contains n elements $\{b_1, b_2, \dots, b_n\}$ with $n > 1$ our method still applies in this case and setting $i(x) = (x_1, x_2, \dots, x_n)$ where $x_i = F_i(x)$ for $i = 1, 2, \dots, n$ we observe that $i: X \rightarrow I^n$ defines an embedding of (R^+, X, α) in the natural similarity semiflow (R^+, I^n, α_c) on the n -dimensional Euclidean cube I^n defined by $\alpha_c(t,x) = c^t x$ for $x \in I^n$ and $t \geq 0$. It is now obvious that the semiflow (R^+, I^n, α_c) can be embedded in (R^+, H, α_c) by sending $(x_1, x_2, \dots, x_n) \in I^n$ into $(x_1, \frac{1}{2}x_2, \dots, \frac{1}{n}x_n, 0, 0, \dots) \in H$ which concludes our proof in all cases.

Remark 3.1. It is obvious that for any $c \in (0,1)$ the semiflow (R^+, H, α_c) belongs to the class S . Since, on the other hand every semiflow of this class can be embedded in (R^+, H, α_c) it is natural to say that the semiflow (R^+, H, α_c) is a universal semiflow for the class S .

Remark 3.2. It is easy to find examples of semiflows (R^+, X, α) of class S with infinite basis B which can be obviously embedded in (R^+, I^n, α_c) for some $n \geq 1$. (Take e.g. the semiflow (R^+, I^2, α_c) . Its basis B is the boundary of the cube I^2 which is an infinite set.) This means that our method of linearization is not too economical as far as dimension is concerned. It is natural to introduce the concept of "minimal linearization dimension of (R^+, X, α) " as an integer (or ∞) $L(R^+, X, \alpha)$ associated with a given semiflow of class S and defined by $L(R^+, X, \alpha) = \inf\{n: \text{there is } n \geq 1$

such that (R^+, X, α) is embeddable in (R^+, I_n, α_c) for some $c \in (0, 1)$. In the next section we shall prove that $L(R^+, X, \alpha)$ is finite if and only if the dimension of the underlying space is finite.

4. An Alternative Method of Linearization Using the Notion of Parallelizability

In [4] N. P. Bhatia and G. P. Szego introduce the concept of parallelizable flows as follows.

Definition 4.1. A flow (R, Y, β) is called parallelizable if there exists a set $V \subset Y$ and a homeomorphism $h: Y \rightarrow R \times V$ such that $h[\beta(t, v)] = (t, v)$ for every $t \in R$ and $v \in V$. The subset V is called a section of (R, Y, β) .

Lemma 4.1. Assume that (R, Y_1, β_1) and (R, Y_2, β_2) are parallelizable flows with the corresponding sections V_1 and V_2 , and assume further that there exists a topological embedding $i: V_1 \rightarrow V_2$.

Then the flow (R, Y_1, β_1) can be embedded in the flow (R, Y_2, β_2) .

Proof. This is obvious if we consider the composition of mappings:

$$Y_1 \ni Y_1 = \beta_1(t, v_1) \rightarrow (t, v_1) \rightarrow (t, i(v_1)) \rightarrow \beta_2[t, i(v_1)] \in Y_2.$$

Lemma 4.2. Let (R, Y^*, β) be the flow obtained from the flow (R, Y, β) constructed in Theorem (2.4) by deleting the point of equilibrium from it, i.e., $Y^* = Y - \{a\}$.

Then the flow (R, Y^*, β) is parallelizable.

Proof. It is obvious that the unit sphere $V = \{y: y \in Y \text{ and } d^*(a, y) = 1\}$ in (R, Y, β) is a section of (R, Y^*, β) .

Theorem 4.3. Given $c \in (0,1)$, any semiflow (R^+, X, α) of class S can be embedded in the flow (R, l_2, α_c) .

Proof. The space Y of the flow (R, Y, β) of Theorem 2.4 is obviously separable since Y is defined as $R \times B \cup \{a\}$. Therefore the unit sphere $V \subset Y$ is also separable which implies that V can be topologically embedded in l_2 . On the other hand it is known that l_2 is homeomorphic to the unit sphere $S^\infty = \{x: \|x\| = 1\}$ of l_2 which in turn implies that there is a topological embedding $i: V \rightarrow S^\infty$ of V in S^∞ .

Denoting by l_2^* the deleted Hilbert space l_2 , i.e., $l_2^* = l_2 - \{0\}$, we conclude from Lemma 4.1 and Lemma 4.2 that the flow (R, Y^*, β) can be embedded in the flow (R, l_2^*, α_c) . It is now clear that any such embedding can be extended to that of (R, Y, β) in (R, l_2, α_c) by sending a to $0 \in l_2$. Since the semiflow (R^+, X, α) is embedded in (R, Y, β) our theorem follows.

Corollary. The minimal linearization dimension $L(R^+, X, \alpha)$ of a semiflow (R^+, X, α) of class S is finite if and only if the dimension of X is finite.

Proof. It is obvious that if $L(R^+, X, \alpha)$ is finite then X has a finite dimension. If on the other hand X is finite dimensional, say $\dim X = n$, then the space Y of Theorem 2.4 has also dimension n since $Y = \cup\{\beta(t, X): t \leq 0\}$. Thus, the dimension of the set V introduced in Lemma 4.2 is of dimension $\leq n$. Consequently V can be embedded in the Euclidean space E^{2n+1} or in the boundary of the $(2n+2)$ -dimensional Euclidean cube I^{2n+2} . Theorem 4.3 trivially modified implies our statement.

Concluding Remark. The method of linearization used in this section applied to semiflows (R^+, X, α) of class S with $\dim(X) = \infty$ does not furnish an embedding of (R^+, X, α) in (R^+, H, α_c) but only in the flow (R, l_2, α_c) . So far we do not know whether this can be remedied by modifying the method suitably.

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