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In [1] J. Ford and J. W. Rogers defined a generalization of the notion of near homeomorphism called refinable map. A map r is *refinable* if and only if r is the uniform limit of ε-maps for every $\varepsilon > 0$. Recall that $f: X \twoheadrightarrow Y$ mapping X onto Y is called an *ε-map* if $\operatorname{diam}(f^{-1}(y)) < \varepsilon$ for each $y \in Y$. (See [4] for a discussion of the usefulness of this concept.) Ford and Rogers proved that if $r: S^2 \to Y$, then r is refinable if and only if it is a near homeomorphism. In this note we show that this result remains true if S^2 is replaced by any closed 2-manifold.

Definition 1. X is said to be Y-*like* if, for each $\varepsilon > 0$, there is an ε-map of X onto Y. If X is Y-like and Y is X-like, then X and Y are said to be *quasi homeomorphic*.

Definition 2. A *generalized cactoid* is a Peano space whose every maximal cyclic element (see [6]) is a closed 2-manifold and only a finite number of these are different from 2-spheres. A *mantoid* is a monotone continuous image of a closed 2-manifold.

Lemma. [5, p. 854]. *The class of mantoids is the class of Peano spaces each of which can be obtained from a generalized cactoid by making a finite number of* 2*-point identifications.*

Theorem. If r *maps the closed* 2*-manifold* M *onto* Y, *then* r *is refinable if and only if* r *is a near homeomorphism.*

Proof. Y is locally connected since M is. So r is monotone by [1, Corollary 1.2] and, therefore, Y is a mantoid. Thus by [5, p. 854] there is a generalized cactoid C and a finite number of 2-point identifications $\phi_1, \phi_2, \cdots, \phi_n$ such that

$$\phi_n \cdots \phi_2 \phi_1(C) = Y.$$

By [1, Corollary 3.2] M and Y are quasi homeomorphic. So dim Y = 2 by [3, p. 64]. Now the singular cohomology of M,

$$H^2(M) = \begin{cases} Z, & \text{if } M \text{ is orientable} \\ Z_2, & \text{if } M \text{ is non-orientable} \end{cases}.$$

So rank $H^2(M) \leq 1$. Since Y is M-like by [4] $Y = \varprojlim\{Y_i, f_i\}$ where each Y_i is a copy of M and $f_i: Y_{i+1} \to Y_i$. So by the continuity of Čech cohomology we have $\check{H}^2(Y) = \varinjlim\{H^2(Y_i), f_i^\#\}$. So rank $\check{H}^2(Y) \leq \sup\{\text{rank } H^2(Y_i)\} = 1$. Therefore, Y has at most one maximal cyclic element which is a closed 2-manifold. Thus C contains at most one (and therefore exactly one) closed 2-manifold. Hence C is an ANR and so therefore is Y. So Y is a 2-dimensional ANR which is M-like. Hence by [2] or [4] Y and M are homeomorphic. So r: M → Y is a non-constant monotone map of a close 2-manifold to a copy of itself. Thus by [7] r is a near homeomorphism.

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