TOPOLOGY PROCEEDINGS
Volume 2, 1977
Pages 265–280
http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN:	0146-4124

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The paper is organized into three sections. In Section 1 we utilize a proof by E. Michael of the Arens-Eells Embedding Theorem to show that if every continuous function from a subspace S of a topological space X into a complete locally convex topological vector space extends to X, then (X,S) satisfies a property relating to the simultaneous order-preserving extension of the bounded continuous pseudometrics on S.

Section 2 introduces various related 'Dugundji-type' extension properties. Results are given showing the relationship of these properties to various concepts involving the simultaneous linear extension of functions. Section 3 consists of examples.

Section 1

The Arens-Eells Embedding Theorem [2] states that every metric space (X,d) can be embedded isometrically as a closed, linearly independent subset of a normed linear space. Michael's proof [12] involves choosing a metric space (Y,d) containing (X,d) and with a point $y_o \in Y-X$. Lip(Y) = $\{f: Y \to R: f(y_o) = 0$ and for some $K \geq 0$, $|f(x) - f(y)| \leq K\, d(x,y)$ for all $x,y \in Y\}$. With $||f||$ equal to the infimum of the K's that work, Lip(Y) becomes a Banach space, and its dual E is also a Banach space. The map $h: X \to E$ is defined by $h(x)(f) = f(x)$ for all $f \in$ Lip(Y). It is then shown that h is an isometry and if $x_1,\cdots,x_n$ are distinct elements of X, then $\{h(x_1),\cdots,h(x_n)\}$ forms a linearly independent set in E. If d is a pseudometric rather than a metric, the following modifications must be made. Let (Y,d) be a pseudometric space containing X with $y_o \in Y-X$ such that $d(y_o,x) > 0$ for all $x \in X$. Let (Y^*,d^*) be the metric space formed from (Y,d) in the usual way. Observe that y_o is not identified with any point of X. Let E be the dual of Lip(Y*) and h be defined as above. Then h is still an isometry but not necessarily 1-1, that is $h(x) = h(y)$ iff $d(x,y) = 0$. If $x_1,\cdots,x_n$ are elements of X such that $d(x_i,x_j) > 0$ for $i \neq j$, then $\{h(x_1),\cdots h(x_n)\}$ forms a linearly independent set in E.

For the remainder of the paper, S will denote a subspace of a topological space X. No separation axioms will be assumed unless stated. All functions and pseudometrics will be continuous. We use the abbreviation l.c.s. to refer to a locally convex topological vector space. Recall that X is P-*embedded* in X if every pseudometric on S can be extended to a pseudometric on X. The following facts are known: (α) S is P-embedded in X iff every bounded pseudometric on S extends to a pseudometric on X ([1], p. 178); (β) S is P-embedded in X iff every function from S to a Fréchet space (complete metrizable l.c.s.) extends to X ([1], p. 227); (γ) X is collectionwise normal iff every closed subset is P-embedded ([1], p. 189); (δ) If υX exists and has non-measurable cardinal, then X is P-embedded in υX ([1], p. 187). $\mathcal{P}^*(X)$ will denote the collection of bounded pseudometrics on X. If $\mathcal{D}$ is a collection of bounded pseudometrics on S a function Φ: $\mathcal{D} \to \mathcal{P}^*(X)$ will be called an *order-preserving extender* if (1) $\Phi(d)$ is an extension of d for all $d \in \mathcal{D}$, and (2) if

$d_1 \leq d_2$, where d_1 and d_2 are in $\mathcal{D}$, then $\Phi(d_1) \leq \Phi(d_2)$.

Theorem 1. *Let* (X,S) *have the property that every function from* S *to a complete l.c.s. extends to* X. *Then there exists an order-preserving extender from* $\mathcal{P}^*(S)$ *to* $\mathcal{P}^*(X)$.

Lemma. Let x_o *be a point not in* S. *There exists a family* $\mathcal{D} = \{d^*\colon d \in \mathcal{P}^*(S)\}$ *of pseudometrics on* $Y = S \cup \{x_o\}$ *such that* d^* *extends* d *and* $d \leq e$ *implies* $d^* \leq e^*$.

Proof. For $d \in \mathcal{P}^*(S)$, let $M = \sup\{d(x,y)\colon x,y \in S\}$ and define $d^*(x,x_o) = M$ for all $x \in S$.

To prove the theorem, observe that the identically zero pseudometric on S will be mapped to the identically zero pseudometric on X, so we will assume it is excluded from $\mathcal{P}^*(S)$. Let Y and $\mathcal{D}$ be as in the Lemma. For each $d \in \mathcal{P}^*(S)$, let Lip_d denote Lip(Y*), where Y* is the metric space associated with (Y,d*). Let E_d denote its dual and h_d the map of Michael from (S,d) into E_d. Let B_d denote the closed span of $h_d(S)$. Suppose $d \leq e$. We will define a linear norm-decreasing mapping g_e^d from B_e to B_d such that the diagram below commutes (where i is the inclusion map):

$$\begin{tikzcd} (S,d) \arrow[r, "h_d"] & B_d \\ (S,e) \arrow[u, "i"] \arrow[r, "h_e"] & B_e \arrow[u, "g_e^d"'] \end{tikzcd}$$

Define $g_e^d\,(h_e(x)) = h_d(x)$ and observe that g_e^d is a well-defined map from $h_e(S)$ into B_d. Extend this map linearly to the span of $h_e(S)$ in B_e. The fact that this extension is well-defined follows from the fact that h_e takes finite sets of points in (S,e) (with distinct pairs of points having positive distance) to a linearly independent set in B_e. We now have g_e^d: span $h_e(S) \to B_d$ and we claim that g_e^d is a norm-decreasing (and hence uniformly continuous) map. This follows from the fact that the unit ball of Lip_d is contained in the unit ball of Lip_e. Since B_d is complete, we may extend g_e^d to a norm-decreasing linear map $g_e^d\colon B_e \to B_d$ and the commutativity of the above diagram is clear.

Let L be the projective limit $\varprojlim g_e^d B_e$, and observe [14] that L is a complete l.c.s. Let $g\colon S \to L$ be defined by $g(x)_d = h_d(x)$, and let g^* denote an extension of g to X. For each $d \in \mathcal{P}^*(S)$, define $\Phi(d)(x,y) = \| g^*(x)_d - g^*(y)_d\|$. Since h_d is an isometry, it is clear that $\Phi(d)$ extends d. Let $d \le e$, then $\Phi(d)(x,y) = \| g_e^d(g^*(x))_e - g_e^d(g^*(y))_e\| = \| g_e^d(g^*(x)_e - g^*(y)_e)\| \le \| g^*(x)_e - g^*(y)_e\| = \Phi(e)(x,y)$. Finally, define $\Psi\colon \mathcal{P}^*(S)$ to $\mathcal{P}^*(X)$ by $\Psi(d) = \Phi(d) \wedge \sup\{d(x,y)\colon x,y \in S\}$.

Using (α), (β), and the fact that the projective limit of a countable family of Banach spaces is a Fréchet space, we have:

Corollary. S *is* P-*embedded in* X *iff there exists an order-preserving extender from any countable family of bounded pseudometrics on* S *to* $\mathcal{P}^*(X)$.

Lutzer and Przymusiński ([10], [11]) have obtained some results relating to the simultaneous extension of pseudometrics. They showed that if S is P^{γ}-embedded in X, then there exists a continuous extender from $\mathcal{P}^*_\gamma(S)$ (the bounded γ-separable pseudometrics on S) to $\mathcal{P}^*_\gamma(X)$, where $\mathcal{P}^*(S)$ is regarded as a subset of $C^*(S \times S)$, with the sup norm topology.

Section 2

To understand how the properties mentioned in the theorem relate to other 'Dugundji type' extension properties, we will consider the following embeddings. If every function from S to a convex subset K of a l.c.s. L extends to X with values in K, we say S is D-*embedded* in X. If L is complete and K closed, S is D*-*embedded* in X. If the extension is only required to go into L (not necessarily complete) we say S is L-*embedded* in X, and if this holds for complete L, then S is CL-*embedded* in X. If the first property holds for K metrizable, then S is M-*embedded* in X, and if the third property holds for L a normed linear space, then S is M*-*embedded* in X. The following relationships hold:

$$\begin{tikzcd} & D^* \arrow[dr, Rightarrow] & \\ D \arrow[ur, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & L \arrow[r, Rightarrow] \arrow[d, Rightarrow] & CL \arrow[d, Rightarrow] \\ M \arrow[r, Leftrightarrow] & M^* \arrow[r, Rightarrow] & P \end{tikzcd} \qquad (*)$$

The only implication that is not obvious is M* implies M. That is shown in Theorem 1 of [15], where M-embedding was introduced and extensively considered. This section will explore the inter-relationships among these embeddings and also attempt to clarify their relation to properties involving the simultaneous linear extension of functions. In Section 3 material developed in this section will be used to show that none of the implications (except M $\Longleftrightarrow$ M*) in the diagram (*) is reversible.

Proposition 1. *The following are equivalent:*

(1) S *is* CL-*embedded in* X.

(2) S *is* P-*embedded in* X *and* $\mathrm{cl}S$ *is* CL-*embedded in* X.

(3) S *is* P-*embedded in* $\mathrm{cl}S$ *and* $\mathrm{cl}S$ *is* CL-*embedded in* X.

Proof. 1) implies 2) implies 3) is obvious. To show that 3) implies 1) let f be a function from S to a complete l.c.s. L. It is clearly only necessary to lift f to $\mathrm{cl}S$. L may be regarded as a closed subspace of a product $\pi\{B_\alpha: \alpha \in A\}$ of Banach spaces. For each α, let f^*_α denote an extension of $p_\alpha \circ f$ to $\mathrm{cl}\, S$. Define f^* from $\mathrm{cl}S$ to L by $f^*(x)_\alpha = f^*_\alpha(x)$. To see that $f^*(x)$ is in L, recall that $f^*(\mathrm{cl}S) \subset \mathrm{cl}(f(S)) \subset L$.

In a similar fashion we obtain:

Proposition 2. *The following are equivalent:*

(1) S *is* D^*-*embedded in* X;

(2) S *is* P-*embedded in* X *and* $\mathrm{cl}S$ *is* D^*-*embedded in* X;

(3) S *is* P-*embedded in* $\mathrm{cl}S$ *and* $\mathrm{cl}S$ *is* D^*-*embedded in* X.

Corollary. *(1) If* S *is dense and* P-*embedded in* X, *then* S *is* D^*-*embedded in* X.

(2) If υX *exists and has non-measurable cardinal, then* X *is* D^*-*embedded in* υX.

Proof. (1) is obvious. For (2), use fact (δ) of Section 1.

The following proposition involves an embedding property that is between L- and CL-embedding.

Proposition 3. *If* υX *has non-measurable cardinal, then every continuous function from* X *to a projective limit of* *normed linear spaces (or metrizable l.c.s.) extends to* υX.

Proof. Let f be a function from X into $L = \varprojlim g_{\alpha\beta}E_\beta$, where each E_α is a normed linear space or each E_α is a metrizable l.c.s. Since X is M-embedded in υX [15], each $p_\alpha \circ f$ has a (unique) extension to $f_\alpha^*: \upsilon X \to E_\alpha$. Defining $f^*: \upsilon X \to L$ by $f^*(x)_\alpha = f_\alpha^*(x)$ gives the result.

For a topological space X and a l.c.s. L, C(X,L) will denote the set of all continuous functions from X to L. By a *simultaneous linear extender* (s.l.e.) from C(S,L) to C(X,L) we mean a linear function $\Psi: C(S,L) \to C(X,L)$ such that $\Psi(f)|S = f$ for all $f \in C(S,L)$. Note that we can further specialize this notion by requiring Ψ to be continuous with respect to various topologies that can be placed on C(X,L). For example, this space can be equipped with the topology of uniform convergence ($C_u(X,L)$), the topology of compact convergence ($C_c(X,L)$), or the topology of pointwise convergence ($C_p(X,L)$). Thus a s.l.e. $\Psi: C_u(S,L) \to C_u(X,L)$ would be a s.l.e. that is continuous with respect to the topology of uniform convergence on both function spaces. The following theorem relates the embedding concepts D, D*, L, and CL to the s.l.e. concept just introduced.

Theorem 2. *Let* S *be a subspace of a topological space* X.

a) S *is* D-*embedded in* X *iff given a l.c.s.* L, *there exists a s.l.e.* $\Psi: C_u(S,L) \to C_u(X,L)$ *such that* $\Psi(f)$ *is contained in the convex hull of* f(S) *for all* $f \in C(S,L)$.

b) S *is* D*-*embedded in* X *iff given a complete l.c.s.* L, *there exists a s.l.e.* $\Psi: C_u(S,L) \to C_u(X,L)$ *such that* $\Psi(f)$ *is* *contained in the closed convex hull of* f(S) *for all* $f \in C(S,L)$.

c) S *is* L-*embedded in* X *iff given a l.c.s.* L, *there exists a s.l.e.* $\Psi\colon C_p(S,L) \to C_p(X,L)$ *such that* $\Psi(f)$ *is contained in the span of* f(S) *for all* $f \in C(S,L)$.

d) S *is* CL-*embedded in* X *iff given a complete l.c.s.* L, *there exists a s.l.e.* $\Psi\colon C(S,L) \to C(X,L)$ *such that* $\Psi(f)$ *is contained in the closed span of* f(S) *for all* $f \in C(S,L)$.

Proof. Observe that in each case sufficiency is clear. To show necessity, let L be a l.c.s. (complete for proof of b) and d) and let E be a product of copies L_f of L, one for each $f \in C(S,L)$. Let ϕ be the canonical map from S into E and let ϕ^* denote an extension of ϕ to X. In the case of a) (b), we assume $\phi^*(X)$ is contained in the (closed) convex hull of $\phi(S)$. In the case of c) (d), we assume $\phi^*(X)$ is contained in the (closed) span of $\phi(S)$. Defining $\Psi(f)(x) = \phi(x)_f$, one checks that all properties except the continuity of Ψ are satisfied. Since there is no continuity requirement in d), the proof of d) is completed.

To show the continuity of Ψ in a), fix $f_o \in C(S,L)$, p a seminorm on L, and $\varepsilon > 0$. Assume $p(f(x) - f_o(x)) < \varepsilon$ for all $x \in S$. Fix $x_o \in X$ and assume $\phi^*(x_o) = \sum_{i=1}^n \alpha_i \phi(x_i)$ is a convex combination of elements in $\phi(S)$. Then $p(\Psi(f)(x_o) - \Psi(f_o)(x_o)) = p(\sum \alpha_i f(x_i) - \sum \alpha_i f_o(x_i)) \leq \sum \alpha_i p(f(x_i) - f_o(x_i)) < \varepsilon$.

To show the continuity of Ψ in b), fix f_o, p, and $\varepsilon > 0$ as above and assume $p(f(x) - f_o(x)) < \varepsilon/3$ for all $x \in S$. Fix $x_o \in X$. Since $\phi^*(x_o)$ is in the closed convex hull of $\phi(S)$ and since E has the product topology, there exists a convex combination $\sum \alpha_i \phi(x_i)$ of elements of $\phi(S)$ such that $p(\sum \alpha_i f(x_i) - \phi^*(x_o)_f) < \varepsilon/3$ and $p(\sum \alpha_i f_o(x_i) - \phi^*(x_o)_{f_o}) < \varepsilon/3$. Hence $p(\Psi(f)(x_o) - \Psi(f_o)(x_o)) \leq p(\Psi(f)(x_o) - \sum \alpha_i f(x_i)) + p(\sum \alpha_i f(x_i) - \sum \alpha_i f_o(x_i)) + p(\sum \alpha_i f_o(x_i) - \Psi(f_o)(x_o)) < \varepsilon$.

To show the continuity of Ψ in c), fix f_o, p, and $\varepsilon > 0$ as above and fix $x_o \in X$. We will find a finite subset F of S and $\delta > 0$ such that $p(f(x) - f_o(x)) < \delta$ for $x \in F$ implies $p(\Psi(f)(x_o) - \Psi(f_o)(x_o)) < \varepsilon$. Now $\phi^*(x_o) = \sum_{i=1}^n \alpha_i \phi(x_i)$, where $x_i \in S$ for $i = 1,\cdots,n$. If $\phi^*(x_o) = 0$, choose F to be some singleton subset of S and choose $\delta = 1$. If $\phi^*(x_o) \neq 0$, we assume $\alpha_i \neq 0$ and $x_i \neq x_j$ for $i \neq j$. Let $F = \{x_i: i = 1, \cdots,n\}$ and $\delta = \frac{\varepsilon}{\sum|\alpha_i|}$. Then if $p(f(x_i) - f_o(x_i)) < \delta$, we have $p(\Psi(f)(x_o) - \Psi(f_o)(x_o)) = p(\sum \alpha_i(f(x_i) - f_o(x_i)) < \varepsilon$.

Corollary 1. *a) If* S *is* L-*embedded in a* $T_{3\frac{1}{2}}$ *space* X, *then* S *is closed in* X.

b) If X *is a* $T_{3\frac{1}{2}}$ *non-realcompact space, then* X *is never* L-*embedded in* υX.

Proof. b) follows from a). To show a), suppose S is L-embedded in X and let $x_o \in \text{cl}S - S$. By c) of Theorem 2, there exists a s.l.e. $\Psi: C_p(S) \to C_p(X)$. Fix 1_S, F a finite subset of S and $\varepsilon > 0$. Choose a continuous function f on X such that $f(x_o) = 1 + \varepsilon$ and $f(x) = 1$ for all $x \in F$. This leads to a contradiction.

Note that the same trick will work to show there exists no s.l.e. from $C_c(X)$ to $C_c(\upsilon X)$. Now let X be a $T_{3\frac{1}{2}}$ non-realcompact space such that υX has non-measurable cardinality.

By the corollary to Proposition 2, we see that X is D*-embedded in υX, but by Corollary 1 above, X is not L-embedded in υX. Proposition 3 shows us that $(\upsilon X, X)$ actually satisfies an embedding property between L- and CL-embedding. Let us now relate our embeddings to the order-preserving extension of real-valued continuous functions.

An extender Ψ from C(S) to C(X) is a *(strict) monotone extender* if $f \leq g$ implies $\Psi(f) \leq \Psi(g)$ ($f < g$ implies $\Psi(f) < \Psi(g)$). The proof of the following corollary is based on the same idea as the proof of Theorem 2.

Corollary 2. *If* S *is* D**-embedded (*D*-embedded) in* X, *there exists a simultaneous linear (strict) monotone extender from* $C_u(S)$ *to* $C_u(X)$.

We now prove a proposition closely related to Theorem 2. It answers a question of Dave Lutzer. Let $\mathcal{S}$ be a countable collection of continuous functions on S with values in a Banach space B, and let $\mathcal{S}^*$ denote the span of $\mathcal{S}$ in C(S,B).

Proposition 4. *Let* S *be a subspace of a topological space* X.

a) S *is* P-*embedded in* X *iff given any Banach space* B *and any* $\mathcal{S}$ *as above, there exists a s.l.e.* $\Psi: \mathcal{S}^* \to C(X,B)$ *such that* $\Psi(f)$ *is contained in the closed convex hull of* f(S) *for all* $f \in \mathcal{S}^*$ *and such that* Ψ *is continuous with respect to the topology* u.

b) S *is* M-*embedded in* X *iff given any countable collection* $\mathcal{S}$ *of functions from* S *into a normed linear space* L, *there* *exists a simultaneous extender* $\Psi: \mathcal{S} \to C(X,L)$ *such that* $\Psi(f)$ *is contained in the convex hull of* $f(S)$ *for all* $f \in \mathcal{S}$ *and such that* Ψ *is continuous when both spaces have the* u *topology or both have the* p *topology.*

Proof. Sufficiency in a) is clear. To show necessity, let $\hat{\mathcal{S}}$ denote all linear combinations of functions in $\mathcal{S}$ with rational coefficients. Then $\hat{\mathcal{S}}$ is countable and we can form the product E of copies B_f of B, one for each $f \in \hat{\mathcal{S}}$. Since E is a Fréchet space, the canonical map ϕ from S into E extends to ϕ^* on X with $\phi^*(X)$ contained in the closed convex hull of $\phi(S)$, ([1], p. 277). Defining $\Psi(f)(x) = \phi^*(x)_f$, we get an extender from $\hat{\mathcal{S}}$ to $C(X,B)$. Considering $\hat{\mathcal{S}}$ as a subset of $C_u(S,B)$ we see (similar to the proof of b) Theorem 2) that Ψ is uniformly continuous. Hence Ψ can be lifted to the closure of $\hat{\mathcal{S}}$ (and this contains $\mathcal{S}^*$). Hence we get a $\Psi^*: \mathcal{S}_u^* \to C_u(X,B)$. One checks that Ψ^* is a linear extender and $\Psi^*(f)(X)$ is contained in the closed convex hull of $f(S)$ for all $f \in \mathcal{S}^*$.

To prove b), observe that sufficiency is clear. To show necessity, form the product E of copies L_f, one for each $f \in \mathcal{S}$ and let ϕ be the canonical map from S into E. Note that E is a metrizable l.c.s. By [15] there exists an extension ϕ^* of ϕ to X with $\phi^*(X)$ contained in the convex hull of $\phi(S)$. The proof is now similar to that of Theorem 2.

Note that in a) Ψ cannot always be assumed to be continuous with respect to the topologies of compact or pointwise convergence, as Heath has shown (p. 28 [9]) that there exists a countable subset $\mathcal{S}$ of $C^*(Q)$ such that no extender (linear or not) from $\mathcal{S}$ to $C^*(X)$, where X is the Michael line, is continuous with respect to either of these topologies. Of course, this implies that Q is not M-embedded in X, a fact we will observe in Section 3.

We now relate the property of D*-embedding to a simultaneous extension property considered in [4]. van Douwen defines a space X to be D_c^*, where $c \geq 1$ is a real number, if for each closed subspace F of X, there is a s.l.e. Ψ from $C^*(F)$ to $C^*(X)$ such that $\|\Psi(f)\| \leq c\ \|f\|$ for each $f \in C^*(F)$. He shows (3.4, p. 304) that if X is a D_1^* space, then X is hereditarily collectionwise normal.

By the proof method of Theorem 2, we obtain:

Proposition 5. *If* S *is* D*-*embedded in* X, *there exists a norm-preserving s.l.e. from* $C^*(S)$ *to* $C^*(X)$.

Section 3

Let us now use the results in Section 2 to show that none of the implications (except $M \Longleftrightarrow M^*$) in the diagram (*) are reversible.

First, let X be a $T_{3\frac{1}{2}}$ non-realcompact space such that υX has non-measurable cardinality. By the comments following Corollary 1 of Theorem 2, we know that X is D*-embedded (and hence CL-embedded) in υX, but it is not L-embedded. Since it is known [15], that X is M-embedded in υX, this shows the non-reversibility of the implications $L \Rightarrow CL$, $L \Rightarrow M^*$, and $D \Rightarrow D^*$.

Heath, Lutzer, and Zenor [7] have shown that there is no s.l.e. from $C_p(\beta N - N)$ to $C_p(\beta N)$. Theorem 2 (c) then implies that $\beta N - N$ is not L-embedded in βN. Since it was shown in [15] that every embedding of a compact space in a $T_{3\frac{1}{2}}$ space is an M-embedding, this produces another example of an M-embedded, non-L-embedded subspace.

Yet another example is essentially contained in Proposition 6.1 of [13]. Let H be a Hilbert space whose orthonormal dimension is the continum and let S be the unit sphere of H in the weak topology. If we let X be the Cartesian product of continum many closed unit intervals, then Michael shows that S is homeomorphic to a closed, convex subset of X. It is known that X is separable, and Michael shows that H with the weak topology is not separable. Let $i: S \to H$ be the identity map, where both S and H have the weak topology. Observe that H with the weak topology is a l.c.s. If i extended to a continuous function i^* on X, then we would have $S \subset i^*(X) \subset H$, which would imply that H is separable, a contradiction. Since S is compact, this produces another example of an M-embedded, non-L-embedded subspace.

Each of the above examples also shows the non-reversibility of the implication $D \Rightarrow M$. Now let X be any compact Hausdorff, non-hereditarily collectionwise normal space. By Proposition 5 and the comments directly preceding it, we know that X must contain a closed subset F that is not D^*-embedded in X. This produces an example of an M-embedded, non-D^*-embedded subspace.

An example due to van Douwen ([4], p. 311) will be used to give an example of an L-embedded, non-D-embedded subspace. Let A_i be a discrete space of cardinality $\aleph_i$ and let $C_i = A_i \cup \{p_i\}$ be the one-point compactification of A_i for $i = 0,1$. Let $X = C_0 \times C_1$, $S = (\{p_0\} \times C_1) \cup (C_0 \times \{p_1\})$ and let f be a function from S into a l.c.s. L. Define an extension f^* on $X - S$ by $f^*(x_o,x_1) = f(x_o,p_1) + f(p_o,x_1) - f(p_o,p_1)$. One can show that f^* is continuous, hence S is L-embedded in X. It follows from van Douwen's results and Proposition 5 that S is not D*-embedded (and hence not D-embedded) in X. This shows the non-reversibility of the implications $D \Rightarrow L$ and $D^* \Rightarrow CL$.

It remains to show the non-reversibility of $M \Rightarrow P$ and $CL \Rightarrow P$. Let X denote the Michael line with Q denoting the rationals. In [15], it is shown that Q is P-embedded in X but not M-embedded. The author would like to express her appreciation to E. Michael for communicating the following proof that Q is not CL-embedded in X.

To show this, let P denote the irrationals with their usual topology, and define $f: Q \times P \to R$ by $f(x,y) = (y-x)^{-1}$. We first show that there is no continuous extension of f to $X \times P$. Suppose g were such an extension. For $n \in N$, let $H_n = \{x \in X-Q: g(x,y) < n \text{ whenever } |x-y| < 1/n\}$. One can show that $X - Q = \cup\{H_n: n \in N\}$. Since Q is not a G_δ in X, we must have $\overline{H}_k \cap Q \neq \emptyset$ for some k.

Pick $x \in \overline{H}_k \cap Q$ and then pick $y \in P$ such that $0 < y-x < 1/2k$. Then $g(x,y) > 2k$. Since g is continuous, there exist neighborhoods U of x and V of y such that $(x',y') \in U \times V$ implies $g(x',y') > k$. Since $x \in \overline{H}_k$, we can choose $x' \in H_k \cap U$ such that $|x-x'| < 1/2k$. Hence $g(x',y) > k$. But $|x'-y| \leq |x'-x| + |x-y| < 1/k$. Therefore, since $x' \in H_k$, we must have $g(x',y) < k$, a contradiction.

Now let $L = C_c(P)$ and define $h: Q \to L$ by $h(x)(y) = f(x,y)$. One can show that h is continuous and L is complete.

Suppose h extended to $h^*: X \to L$. Defining $g: X \times P \to R$ by $g(x,y) = h^*(x)(y)$ would give a continuous extension of f, contrary to the above result.

The author would like to thank the referee for various suggestions leading to a clearer presentation of the results in this paper.

Comments and Open Questions

1. Is the converse of Theorem 1 true?
2. If not, find a P-embedded subset that fails to satisfy the conclusion of Theorem 1.
3. From Theorem 2 it is clear that if S is D-embedded in X, then there exists a s.l.e. from $C_u(S)$ to $C_u(X)$ and from $C_p(S)$ to $C_p(X)$. Must there exist a s.l.e. from $C_c(S)$ to $C_c(X)$?
4. Give characterizations (similar to those known for P- and M-embedding) for the other embeddings introduced in Section 2.

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