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# Research Announcement: MAXIMAL CONNECTED HAUSDORFF TOPOLOGIES

by

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| **Web:** | http://topology.auburn.edu/tp/ |
| **Mail:** | Topology Proceedings<br>Department of Mathematics & Statistics<br>Auburn University, Alabama 36849, USA |
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*Research Announcement*

# MAXIMAL CONNECTED HAUSDORFF TOPOLOGIES

**J. A. Guthrie, H.E. Stone, and M.L. Wage**

1. One of the more interesting problems in the lattice of topologies during the last ten years has been the question [4,3] of the existence of non-trivial Hausdorff topologies maximal with respect to connectedness. Properties of such spaces and partial results have been obtained by Thomas [5], Reynolds [1], and Guthrie and Stone [1,2] among others. The object of this announcement is to outline a solution by constructing a maximal connected expansion of the real line. Such constructions have been obtained jointly by Guthrie and Stone, and independently by Wage. Both will be described briefly, and they will be compared, but details will appear elsewhere. Both examples are modifications of the real line R with the Euclidean topology $\varepsilon$. We conclude with some discussion of the related open questions which remain.

2. It has long been known that expansions of connected topologies obtained by adjoining a filter of dense sets remain connected; hence a necessary condition for a space to be maximally connected is that the space be *submaximal*--every dense set is open. In attempting to get a maximal connected space, it is natural to adjoin sets as far removed from dense sets as possible--in fact, to examine expansions which to not change the dense sets.

A typical such set is the inverse image of an interval about 0 under the topologist's sine curve. In particular, we call a set $S \subseteq R$ *singular* at $x \in S$ if

a) $S - \{x\} \in \varepsilon$

b) $x \in Cl_\varepsilon(S \cap (-\infty,x)) \cap Cl_\varepsilon(S \cap (x,\infty))$.

We call an expansion of $\varepsilon$ a *singular expansion* if every local base at x consists of sets singular at x. If every singular set is open, we call the space *nonsingular*. Since clearly adjoining a singular set preserves connectedness, nonsingularity is a necessary condition for maximal connectedness.

*Theorem* 1. *An expansion of the reals which is connected, nonsingular and submaximal is maximally connected.*

It is easy to get a nonsingular expansion of $\varepsilon$: merely adjoin for each x a maximal filter of sets which are singular at x. The problem is to obtain such a space which is connected. The key to such a construction is the following result about disconnected singular expansions of $I = [0,1]$.

*Theorem* 2. *Let* (A,B) *disconnect a singular expansion* $\sigma$ *of* $(I,\varepsilon)$. *If* $C = I - (Int_\varepsilon A \cup Int_\varepsilon B)$, *then* $(C,\varepsilon)$ *is the Cantor space.*

Each disconnection (A,B) is related to a Cantor set; but there are at most $2^\omega$ of these. Hence for each real x we can associate a copy of the Cantor set $C(A_x,B_x)$ such that x is a non-endpoint of C, and a set $S_x$ singular at x such that $S_x \cap A_x$ and $S_x \cap B_x$ both cluster to x. Refine $\{S_x\}$ to a maximal singular filter at x and adjoin all such to $\varepsilon$. The result must be connected, for each possible disconnection has been defeated in advance.

*Theorem* 3. *There exists a connected nonsingular expansion of* $\varepsilon$*; hence there is a maximal connected expansion of* $\varepsilon$.

3. Wage's construction proceeds in two steps which are analogous to those above. The first of these takes place within the rationals Q, where it is desired to find sets U having the property

$$*)\ \text{If } q \in U \text{ then } (-\infty,q) \cap U \neq \emptyset \neq U \cap (q,\infty).$$

The similarity of this property to the concept of a singular set is clear.

*Theorem* 4. *There exists a topology* $\sigma$ *for* Q *which refines the Euclidean topology* $\varepsilon$ *for* Q *and which is maximal with respect to the property that if* $U \in \sigma$ *then* U *has property* (*).

Now one can inductively order the irrationals $\{x_a\colon a < 2^\omega\}$ and the clopen subsets $\{U_a\colon a < 2^\omega\}$ of $\sigma$ such that for each $a < 2^\omega$, $x_a \in Cl_\varepsilon(Q - U_a) \cap Cl_\varepsilon U_a$. For each $a < 2^\omega$, choose maximal filters $\mathcal{A}_a$ and $\mathcal{B}_a$ such that for $A \in \mathcal{A}_a$ and $B \in \mathcal{B}_a$,

a) $A \subseteq Q \cap (x_a,\infty)$, $B \subseteq Q \cap (-\infty,x_a)$

b) $A,B \in \sigma$

c) $x_a \in Cl_\varepsilon A \cap Cl_\varepsilon B$

d) A and B are separated by $U_a$.

Define a topology $\rho$ for R to have base $\sigma \cup \{A \cup \{x_a\} \cup B\colon a < 2^\omega, A \in \mathcal{A}_a, B \in \mathcal{B}_a\}$.

*Theorem* 5. *The topology* $\rho$ *is maximal connected.*

The inductive ordering above is analogous to the association of reals with Cantor sets in the construction of the connected nonsingular topology of Theorem 3. The basic similarity of the constructions seems clear. The construction of Wage is efficient, but seems to be tied to Q or other countable dense subset. By selecting the proper ultrafilter of dense subsets, the topology constructed from Theorems 3 and 1 can be made to have dispersion character $2^{\omega}$.

4. Now that a maximal connected topology for the reals has been constructed, we call attention to some corollaries and to some of the remaining questions. If a maximally connected topology is put on each line through the origin, and the plane is given the weak topology with respect to these lines, the result is plainly maximally connected. This generalizes easily to the following observation.

*Theorem* 6. *Let* S *be a starlike subset of any Euclidean space. Then* S *admits a maximally connected expansion.*

We might also ask how much more than Hausdorff separation can be obtained. Since our maximal connected topology is finer than $\varepsilon$, we automatically have an example which is Urysohn and functionally Hausdorff.

*Question* 1. *Does there exist a (semi-) regular Hausdorff maximally connected space? finer than the real line?*

Both of these constructions are confined to the reals, and a key step in each case is made possible by the fact that $2^{\omega}$ points are involved. Do maximal connected Hausdorff topologies exist for spaces of other cardinality? In particular,

*Question* 2. *Does there exist a countable maximally connected Hausdorff space?*

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University of Texas at El Paso

El Paso, TX 79968

University of Southwestern Louisiana

Lafayette, Louisiana 70504

Institute for Medicine and Mathematics, Ohio University

Athens, OH 45701

and

Yale University (current)

New Haven, Connecticut 06520