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Research Announcement:

ON EMBEDDINGS OF MANIFOLDS INTO CARTESIAN PRODUCTS OF COMPACTA

by

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ON EMBEDDINGS OF MANIFOLDS INTO CARTESIAN PRODUCTS OF COMPACTA

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All spaces considered in this note are metric and compact. If X and Y are spaces, by an embedding of X in Y, we understand a homeomorphism from X onto a subset of Y. A closed manifold is a (compact metric) connected manifold without boundary. A surface is a 2-dimensional manifold.

Theorem 1. *If* M *is an* n-*dimensional* ($n \geq 2$) *closed manifold whose fundamental group* $\pi_1(M)$ *is finite and if* X *and* Y *are spaces with dim* X = n-1 *and dim* Y = 1, *then there exists no embedding of* M *in the Cartesian product* X×Y.

Corollary 1. *The* n-*sphere* S^n ($n \geq 2$) *cannot be embedded in the product* X×Y *of spaces* X *and* Y *with dim* X = n-1 *and dim* Y = 1.

This corollary is a generalization of a result of K. Borsuk [1] who proved the non-existence of an embedding of the 2-sphere in the product of two one-dimensional spaces. Let us note that Borsuk's result gave a solution to the following problem from dimension theory, due to J. Nagata [2]: Is it true that for every n-dimensional space Z there exist spaces $X_1, X_2, \cdots, X_n$ of dimension at most one such that Z can be embedded in $X_1 \times X_2 \times \cdots \times X_n$?

Problem 1. *Is it true that the* n-*sphere* S^n *cannot be embedded in the product* X×Y *of spaces* X *and* Y *of positive* *dimension and with* $\dim X + \dim Y = n$?

Corollary 2. *The projective plane cannot be embedded in the product of two one-dimensional spaces.*

Example. Let k be a positive integer and let X be the subset of the plane E^2 defined by $X = \{0,1\} \times [0,k] \cup [0,1] \times \{0,1,2,\cdots,k\}$. The space X is one-dimensional and the product $X \times X$ contains an orientable closed surface of genus k.

Problem 2. *Is it true that the orientable closed surfaces of positive genus are the only closed surfaces embeddable in the products of two one-dimensional spaces?*

Theorem 2. *If* M *is a closed* n*-dimensional manifold contained in the product* $X \times S^q$ *of a* p*-dimensional space* X *and the* q*-sphere* S^q, *where* $p + q = n$, $q \geq 1$, *then there exists a subset* A *of* X *such that* $M = A \times S^q$.

In other words, this theorem states that if M is embeddable in $X \times S^q$, then not only S^q is a factor of M, but also every embedding of M in $X \times S^q$ is a "*product-embedding.*"

Corollary 3. *If* X *is a* 1*-dimensional space and if* T *is a closed surface embeddable in* $X \times S^1$, *then* T *is a torus surface and, moreover, every embedding of* T *in* $X \times S^1$ *is a homeomorphism onto* $A \times S^1$, *where* A *is some simple closed curve in* X.

Problem 3. *Suppose that* M *is a closed* (p+q)*-dimensional manifold contained in the product* $X \times N$ *of a* p*-dimensional space* X *and a* q*-dimensional closed manifold* N. *Does there exist a subset* A *of* X *such that* $M = A \times N$?

Problem 4. *Suppose that* T *is a torus surface contained in the product* X×Y *of two one-dimensional spaces* X *and* Y. *Do there exist two simple closed curves* A ⊂ X *and* B ⊂ Y *such that* T = A×B?

Added in proof:

The author was informed recently that, after this announcement was submitted for publication, problem 3 was solved in the affirmative by George Kozlowski, and that problem 1 was solved in the negative by Wlodzimierz Holsztynski and Andrzej Kadlof, independently.

References

1. K. Borsuk, *Remark on the Cartesian product of two 1-dimensional spaces*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 23 (1975), 971-973.
2. J. Nagata, *Modern dimension theory*, Amsterdam, 1965.

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