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# Research Announcement:

# A COMPACT NONMETRIZABLE SPACE $P$ SUCH THAT $P^2$ IS COMPLETELY NORMAL

by

Peter Nyikos

**Topology Proceedings**

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| **Web:** | http://topology.auburn.edu/tp/ |
| **Mail:** | Topology Proceedings<br>Department of Mathematics & Statistics<br>Auburn University, Alabama 36849, USA |
| **E-mail:** | topolog@auburn.edu |
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*Research Announcement*

# A COMPACT NONMETRIZABLE SPACE P SUCH THAT $P^2$ IS COMPLETELY NORMAL

**Peter Nyikos**

In 1948, M. Katětov showed [2] that if P is a compact space such that P × P × P is completely normal, then P is metrizable. [Throughout this announcement, the word "space" will always refer to a Hausdorff space.] Katětov then remarked that he did not know whether complete normality of P × P was enough to give metrizability. I will now give an example of a compact, nonmetrizable space P whose square, assuming Martin's axiom and the negation of the continuum hypothesis [abbreviated MA + ¬CH] is completely normal. I will also indicate why a "naive" example of such a space, if it exists, will be hard to come by.

## 1. The Example

Let I = [0,1]. Let A ∈ I be of cardinality $\aleph_1$. Replace each a of A by two points, $a^+$ and $a^-$, with the convention that $a^- < a^+$; otherwise we keep the usual order of all of $(I-A) \cup A^+ \cup A^- = P$. Since P is complete in the order topology, it is a compact space, and nonmetrizable because it does not have a countable base: every base must include an open set with $a^+$ as its least element, and one with $a^-$ as its greatest, for all a ∈ A.

## 2. Preliminary Lemmas

The following oft-used lemma is utilized in the proof:

*Lemma* 1. *Let* H *and* K *be subsets of a space* X. H *and*

K *can be put into disjoint open subsets whenever there exist countable collections of open sets* $U_n$ *and* $V_n$ *so that* $H \subset \bigcup_{n=1}^{\infty} U_n$, $K \subset \bigcup_{n=1}^{\infty} V_n$ *and* $\overline{U}_n \cap K = \phi$ *and* $\overline{V}_n \cap H = \phi$ *for all* n.

We also use:

*Lemma* 2. [3] [MA + ¬CH] *Let* Y *be a metric space of cardinality* $\aleph_1$. *Every subset of* Y *is an* $F_\sigma$.

We also note that $(A^+)^2$ is homeomorphic to a subspace of the Sorgenfrey plane.

*Lemma* 3. [1] [MA + ¬CH] *If* X *is a subspace of the Sorgenfrey plane, of cardinal* $\aleph_1$, *then* X *is normal.*

*Lemma* 4. *The space* $P^2 - (A^+ \cup A^-)^2$ *is hereditarily Lindelöf.*

Now, although $P^2$ is the union of the perfectly normal (under MA + ¬CH) subspaces $P^2 - (A^+ \cup A^-)^2$, $(A^+)^2$, $(A^-)^2$, $A^+ \times A^-$, and $A^- \times A^+$, the whole space is not perfectly normal; for example, the diagonal is not a $G_\delta$. But under MA + ¬CH it is completely normal.

## 3. Outline of the Proof

Let H and K be subsets of $P^2$ such that $\overline{H} \cap K = \phi$, $H \cap \overline{K} = \phi$. The objective is to find countable collections $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ of open subsets of $P^2$ as in Lemma 1. Using Lemma 4 and regularity of $P^2$, we can get $H - (A^+ \cup A^-)^2$ into a countable union of open sets whose closure misses K. By various symmetry arguments, the proof then boils down to showing that there exist countably many open sets whose closures miss K and whose union contains $H \cap (A^+)^2 = H_1$.

With the help of Lemma 3, we can get $H_1$ into an open subset of $P^2$ whose closure misses $K \cap (A^+)^2$. We will choose all our open sets to be contained in this one. By Lemma 2 [there is a coarser metric topology on $(A^+)^2$] we can let $H_1 = \bigcup_{n=1}^{\infty} F_n$ where each $F_n$ is closed in $(A^+)^2$, and each $x \in F_n$ has a basic first-quadrant neighborhood whose closure misses K and which is a square 1/n on a side. For each $F_n$ we cut up $P^2$ into countably many clopen squares <1/n on a side. It is enough to take care of the points of $F_n$ which lie in any one square S. Attach the basic 1/n-neighborhoods to these points, trimming off the parts sticking out of S. The only possible points of K in the closure of the resulting open set lie on a graph which can be thought of as a monotone function. Moreover, points of K can only lie along straight line segments of the graph, of which there are countably many.

Let $E_n$ stand for the one-sided limit points of $F_n \cap S$ on the graph. We use Lemma 2 again to show $F_n \cap S = \bigcup_{n=1}^{\infty} C_{nm}$ where each $C_{nm}$ is closed in the relative *Euclidean* topology of $E_n \cup (F_n \cap S)$. If we attach the basic 1/n-neighborhoods to the points of $C_{nm}$ and intersect with S, it turns out that the closure of the resulting set misses K.

## 4. What Happens if $2^{\aleph_0} < 2^{\aleph_1}$

The space $P^2$ is *not* completely normal in any model of set theory where $2^{\aleph_0} < 2^{\aleph_1}$. This is because $P^2$ has the uncountable discrete subspace $\{(a^+,a^-) | a \in A\}$ and is separable, and it is well known that:

*Lemma* 5. $[2^{\aleph_0} < 2^{\aleph_1}]$ *Every discrete subspace of a separable, completely normal space is countable.*

This lemma quickly involves us in a famous pair of problems of general topology: whether there exists an S-space (a regular, hereditarily separable space which is not hereditarily Lindelöf) or an L-space (a regular, hereditarily Lindelöf space which is not hereditarily separable). Such spaces have been constructed in some models of set theory; but, in particular, no one has constructed such spaces assuming *only* $2^{\aleph_0} < 2^{\aleph_1}$.

But from Lemma 5 it is only a short step to:

*Theorem* $[2^{\aleph_0} < 2^{\aleph_1}]$ *If* X *is a compact, nonmetrizable space such that* $X^2$ *is completely normal, then at least one of the following is true:*

1. X *is an* L-*space*
2. $X^2$ *is an* S-*space*
3. $X^2$ *contains both an* S-*space and an* L-*space.*

If we combine this theorem with the result that MA + $\neg$CH implies every compact space of countable spread is hereditarily separable, we see that a "naive" example (one whose basic cardinal invariants like density, spread, and hereditary Lindelöf degree do not vary with the model of ZFC used) would have a hereditarily separable square which is not hereditarily Lindelöf (because of not having a $G_\delta$-diagonal). Such a compact space has not yet been constructed in *any* model of set theory!

Even if we restrict ourselves to $2^{\aleph_0} < 2^{\aleph_1}$, we are out of reasonable candidates: Mary Ellen Rudin showed, after I came up with these results, that the square of a Souslin line is never completely normal.

## References

[1] K. Alster and T. Przymusiński, *Normality and Martin's Axiom*, Fund. Math. 91 (1976), 123-131.

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Auburn University

Auburn, Alabama 36830

and

Ohio University Institute for Medicine and Mathematics

Athens, Ohio 45701