Research Announcement:

OPEN RETRACTIONS OF CONTINUA

by

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The origin of this research[1] was the following problem
raised by A. R. Stralka in 1972: Is it true that the only
open monotone retraction of a dendroid onto an arc is the
identity mapping? (Compare [1], p. 266.) A *dendroid* is an
arcwise connected hereditarily unicoherent continuum. By
a *continuum* we mean a connected compact metric space, and a
mapping is meant to be a continuous function. Each dendroid
is a hereditarily decomposable continuum. A non-trivial
open monotone mapping of an irreducible hereditarily unico-
herent continuum onto an arc has been constructed by Knaster
[3], and Dyer [2] has shown that each such a mapping must
possess point-inverses which are indecomposable continua.
Although the solution of Stralka's problem is in the negative
(see 2.1 below), we offer a number of positive results under
some additional conditions imposed upon continua or mappings
(see 1.1-1.6).

A complete version of the present paper wlll be pub-
lished elsewhere. The author wishes to thank Professor A.
Lelek for continuous encouragement and valuable improvements.

1. Theorems

We use a method developed by Dyer [2] to prove the fol-
lowing result.

1.1. *Theorem. Let* r: X → Y *be an open retraction of a*

compact metric space X onto a non-degenerate continuum

$Y \subset X$ *satisfying the conditions:*

  (i) if $C \subset X$ *is a continuum and* $r(C)$ *is non-degenerate,*
      then $C \cap Y \neq \emptyset$,

  (ii) if $y \in Y$ *and* $r^{-1}(y)$ *is non-degenerate, then there*
      exists a continuum $K \subset r^{-1}(y)$ *such that* $y \notin K$ *and*
      the interior of K *in* $r^{-1}(y)$ *is non-empty.*

  Then $X = Y$ *and* z *is the identity mapping.*

For monotone retractions of hereditarily unicoherent
continua, condition (i) is automatically satisfied. Easy
examples show that this condition cannot be omitted in
Theorem 1.1. To see that condition (ii) also cannot be
omitted, a more complicated construction is needed (see
Example 2.1). The next theorem is an analogue of 1.1 for
mappings instead of retractions. Here, however, the assump-
tion that the domain space is hereditarily unicoherent has
to be included (see Example 2.3). We say that a compact
metric space X is *weakly aposyndetic* provided, for each point
$x \in X$, there exists a continuum $K \subset X$ such that $x \notin K$ and
Int $K \neq \emptyset$.

  1.2. *Theorem.* *Let* f: $X \to Y$ *be an open mapping of a*
hereditarily unicoherent compact metric space X onto a non-
degenerate continuum Y satisfying the conditions:

  (I) if $C_1, C_2 \subset X$ *are continua,* $f(C_1)$ *is non-degenerate*
      and $f(C_1) \subset f(C_2)$, *then* $C_1 \cap C_2 \neq \emptyset$,

  (II) if $y \in Y$ *and* $f^{-1}(y)$ *is non-degenerate, then* $f^{-1}(y)$
      is weakly aposyndetic.

  Then f *is a homeomorphism.*

Moreover, condition (I) can be replaced by the assumption that f is monotone.

The following four theorems are analogues of Theorems 1.1 and 1.2 for quasi-interior and locally confluent mappings [6]. We say that a mapping f: X → f(X) is *weakly irreducible* provided C ≠ X implies f(C) ≠ f(X) for each continuum C ⊂ X.

1.3. *Theorem. If f: X → Y is a quasi-interior weakly irreducible mapping of a compact metric space X onto a semi-locally connected continuum Y, then f is monotone and X is a continuum. If, in addition, f is 0-dimensional, then f is a homeomorphism.*

Without assuming that the range continuum is semi-locally connected, Theorem 1.3 is no longer true (see Example 2.4). Also, the condition of f being quasi-interior and weakly irreducible in 1.3 cannot be replaced by that of being open and 0-dimensional or confluent and weakly irreducible (see Examples 2.5 and 2.6).

1.4. *Theorem. If f: X → Y is a locally confluent weakly irreducible mapping of a compact metric space X onto a locally connected continuum Y, then f is monotone and X is a continuum. If, in addition, f is 0-dimensional, then f is a homeomorphism.*

1.5. *Theorem. If r: X → Y is a locally confluent retraction of a compact metric space X onto a non-degenerate continuum Y ⊂ X satisfying condition (i) of Theorem 1.1, then r is monotone and X is a continuum. If, in addition,*

r *is 0-dimensional, then* X = Y *and* r *is the identity mapping.*

1.6. *Theorem. If* f: X → Y *is a locally confluent mapping of a compact metric space* X *onto a non-degenerate continuum* Y *satisfying condition (I) of Theorem 1.2, then* f *is monotone and* X *is a continuum. If, in addition,* f *is 0-dimensional, then* f *is a homeomorphism.*

Clearly, conditions (i) and (I) are essential in Theorem 1.5 and 1.6, respectively.

2. Examples

2.1. *Example. There exists an open monotone retraction* r: X → A *of a dendroid* X ⊂ R³ *onto an arc* A ⊂ X *such that* r *is not the identity mapping.*

During the preparation of this paper,[2] the author has learned that Krasinkiewicz and Minc [4] have independently constructed a similar example solving Stralka's problem. Their construction is, however, different from and less geometrical than ours. We also use Example 2.1 to build another dendroid, as described in 2.2, which provides a solution to a problem raised by A. Lelek (see [5], p. 328).

2.2. *Example. There exists an open path-raising mapping* f: X → Y *of a dendroid* X ⊂ R³ *onto a dendrite* Y *such that* f(C) ≠ Y *for each locally connected continuum* C ⊂ X.

2.3. *Example. There exists an open monotone mapping* f: X → Y *of a 1-dimensional continuum* X *onto an arc* Y *such*

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[2]Presented at the Eleventh Spring Topology Conference, Baton Rouge, Louisiana, on March 12, 1977.

*that both conditions (I) and (II) of Theorem 1.2 are satis-
fied and* f *is not a homeomorphism.*

2.4. *Example. There exists an open weakly irreducible
finite-to-one mapping* f *of the pseudo-arc onto itself such
that* f *is not a homeomorphism.*

The existence of a mapping claimed in 2.4 is the result
of the existence of a continuous involution on the pseudo-
arc.

2.5. *Example. There exists a local homeomorphism* f *of
the circle onto itself such that* f *is not a homeomorphism.*

2.6. *Example. There exists a confluent weakly irreduc-
ible finite-to-one mapping* f: X → Y *of a semi-locally con-
nected continuum X onto a semi-locally connected continuum
Y such that* f *is not quasi-interior (hence* f *is not mono-
tone).*

References

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