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by

H. R. Bennett and D. J. Lutzer

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Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
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1. Introduction

A collection $\mathcal{C}$ of subsets of X is *minimal* or *irreducible* if each $C \in \mathcal{C}$ contains a point x(C) belonging to no other member of $\mathcal{C}$. Thus $\mathcal{C}$ is irreducible if and only if $\cup \mathcal{D} \subsetneq \cup \mathcal{C}$ whenever $\mathcal{D} \subsetneq \mathcal{C}$. (It is crucial to note that $\cup \mathcal{C} = X$ is *not* required in this definition.) A collection which is the union of countably many minimal collections is said to be *σ-minimal*.

Recall that a space X is *quasi-developable* if there is a sequence $\langle \mathcal{G}(n) \rangle$ of open collections (not necessarily covers) in X such that if U is open and $p \in U$ then for some n, $p \in St(p, \mathcal{G}(n)) \subset U$. In [BL] the authors obtained a result from which it follows that every quasi-developable space has a σ-minimal base for its topology. C. E. Aull [Au] initiated the study of σ-minimal bases in their own right; in particular he asked about conditions under which a space with a σ-minimal base must be quasi-developable.

This paper grew out of the surprising observation by Bennett and Berney [BB_1] that the lexicographic square S has a σ-minimal base. That example illustrates how disparate are the notions of a quasi-development and a σ-minimal base. More precisely, the example shows simultaneously that (1) a space can be compact, Hausdorff, have a σ-minimal base and yet may fail to be metrizable; and (2) that the existence of a σ-minimal base is not a closed-hereditary property. Both of these assertions follow from the fact that the Alexandroff "double-arrow" space A (i.e., the top and bottom edges of the square S) is a hereditarily Lindelöf and yet non-metrizable (closed) subspace of S whence A cannot have a σ-minimal base [Au].

In this paper we study the structure of ordered spaces having σ-minimal bases. Recall that a *linearly ordered topological space* (LOTS) is a linearly ordered set equipped with the usual open interval topology. By a *generalized ordered space* (GO space) we mean a linearly ordered set equipped with a T_1-topology having a base whose members are convex. It is known that the class of GO spaces coincides with the class of (closed) subspaces of LOTS [L_1, 2.9]. As usual, we conduct our study in the class of GO spaces whenever possible.

At several points in our paper we will need to invoke theorems from the literature. For the sake of completeness we list them here.

1.1 *Theorem* [EL]. *A* GO *space* X *is* not *hereditarily paracompact if and only if some subspace of* X *is homeomorphic to a stationary subspace of a regular uncountable cardinal.*[1]

Recall that a completely regular space X is a p-*space* [Ar] if there is a sequence $\langle \mathcal{G}(n) \rangle$ of collections of open

[1] A *cardinal* is an initial ordinal and is identified with the set of all ordinals which precede it. Thus we write ω_1 for $[0,\omega_1[$. A cardinal κ is *regular* if it is not the sum of fewer, smaller cardinals. A subset S of a cardinal κ is *stationary* if $S \cap C \neq \emptyset$ whenever C is a cofinal subset of κ which is closed in the usual topology of κ.

subsets of βX (equivalently, in any compactification of X) such that if $p \in X$ then $\cap\{St(p,\mathcal{G}(n)) | n \geq 1\} \subset X$ where $St(p,\mathcal{G}(n)) = \cup\{G \in \mathcal{G}(n) | p \in G\}$.

1.2 *Theorem* [B_1]. *A completely regular space* X *is metrizable if and only if* X *is a quasi-developable paracompact* p-*space.*

1.3 *Theorem* [vW]. *A* LOTS *having a* σ-*discrete dense subspace is a paracompact* p-*space.*

1.4 *Pressing Down Lemma. Let* S *be a stationary subset of a regular uncountable cardinal* κ *and let* $f: S \to \kappa$ *satisfy* $f(x) < x$ *whenever* $x \in S - \{0\}$. *Then for some* $y \in \kappa$, $f^{-1}[\{y\}]$ *is a stationary subset of* κ.

1.5 *Theorem* [B_2]. *A space* X *is metrizable if and only if* X *is collectionwise normal, quasi-developable and has the property that every closed subset of* X *is a* G_δ-*set.*

We will use the symbol $|S|$ to denote the cardinality of a set S. Our notation and terminology for ordered spaces will follow that of [L_1]. For example, we write $]p,q[= \{x \in X | p < x < c\}$ and $]p,\to[= \{x \in X | p < x\}$. Let N denote the set of natural numbers.

2. Ordered Spaces Having σ-Minimal Bases

It is well known that the lexicographic square, cited in the Introduction as being a pathological ordered space having a σ-minimal base, is hereditarily paracompact. That is no accident, as our first result shows.

2.1 *Theorem. Any generalized ordered space having a σ-minimal base is hereditarily paracompact.*

Proof. Let $\mathcal{B} = \cup\{\mathcal{B}(n): n \geq 1\}$ be a σ-minimal base for X. Supposing X is not hereditarily paracompact, it follows from Theorem (1.1) that there is a regular uncountable cardinal κ and a stationary subset S of κ such that S is homeomorphic to a subspace of X. Since the homeomorphism can be taken to be either order-preserving or order-reversing, we may assume that $S \subset X$ and that the ordering which S inherits from X coincides with the ordering which S inherits from κ. We will use small Greek letters to denote points of S. The supremum of S is either a point or a gap in X; in either case, denote it by κ.

Let $T = \{\lambda \in S | \lambda$ is a limit point of $S\}$. Since S is stationary in κ, so is T. For each $\lambda \in T$, let $\hat{\lambda}$ denote the first element of T which is greater than λ. Then for each $n \geq 1$, define $T(n) = \{\lambda \in T |$ for some $B(\lambda) \in \mathcal{B}(n)$, $\lambda \in B(\lambda) \subset]\leftarrow,\hat{\lambda}[\}$. Because $\mathcal{B}$ is a base for X, $T = \cup\{T(n) | n \geq 1\}$. Therefore the set $M = \{m \in N | T(m)$ is stationary$\}$ is nonvoid. For each $m \in M$ define a function $f_m: T(m) \to S$ as follows. For $\lambda \in T(m)$ let $f_m(\lambda)$ be the first element of S satisfying $f_m(\lambda) < \lambda$ and $]f_m(\lambda),\lambda] \subset B(\lambda)$. Since f_m is regressive, the Pressing Down Lemma (1.4) guarantees that for some $\beta(m) \in S$, the set $T'(m) = \{\lambda \in T(m) | f_m(\lambda) = \beta(m)\}$ is stationary in κ. Because κ has uncountable cofinality, there is a $\gamma \in S$ such that $\beta(m) < \gamma < \kappa$ for each $m \in M$.

Let $R = T \cap [\gamma + 1, \kappa)$ and for each $n \geq 1$ let $R(n) = \{\lambda \in R |$ some $C(\lambda) \in \mathcal{B}(n)$ has $\lambda \in C(\lambda) \subset]\gamma,\hat{\lambda}[\}$. Then R is stationary in κ and, because $\mathcal{B}$ is a base for X,

$R = \cup\{R(n) | n \geq 1\}$. Hence for some n_0, $R(n_0)$ is stationary. Because $R(n_0) \subset T(n_0)$, $n_0 \in M$. Fix $\mu \in R(n_0)$ and consider $C(\mu)$. Since $C(\mu) \in \mathcal{B}(n_0)$ it is possible to find a point $p \in C(\mu)$ such that $\text{ord}(p, \mathcal{B}(n_0)) = 1$.

Since $n_0 \in M$, the stationary set $T'(n_0)$ and the ordinal $\beta(n_0)$ exist and we may choose elements $\lambda_1, \lambda_2 \in T'(n_0)$ having $\hat{\mu} < \lambda_1 < \hat{\lambda}_1 < \lambda_2$. Because $\lambda_i \in T'(n_0)$ we have $f_{n_0}(\lambda_i) = \beta(n_0) < \gamma$ for $i = 1,2$ so that $p \in]\gamma, \hat{\mu}[\subset]f_{n_0}(\lambda_i), \lambda_i] \subset B(\lambda_i) \in \mathcal{B}(n_0)$ for $i = 1,2$. And since $\lambda_2 \in B(\lambda_2) - B(\lambda_1)$, $B(\lambda_1)$ and $B(\lambda_2)$ are seen to be two distinct members of $\mathcal{B}(n_0)$ containing p which is impossible because $\text{ord}(p, \mathcal{B}(n_0)) = 1$. That contradiction completes the proof of the theorem.

2.2 *Remarks*. (a) The proof of Theorem (2.2) differs from the usual proofs of paracompactness in ordered spaces using [EL] because the property "X has a σ-minimal base" is *not* closed hereditary. Thus it is *not* sufficient to prove that no stationary set in an uncountable regular cardinal has a σ-minimal base.

(b) There is a covering property related in the usual way to the existence of a σ-minimal base: a space X is *σ-irreducible* if every open cover of X has a σ-irreducible open refinement. An example due to van Douwen [vD] shows that a LOTS can be σ-irreducible and yet may fail to be paracompact (let D be an uncountable linearly ordered set, having a first element, which is discrete in the order topology; then the lexicographic product $\omega_1 \times D$ is the required example.) However it is easy to see that a GO space X is paracompact if and only if every closed subspace of X is σ-irreducible;

one utilizes Theorem (1.1) after proving that no stationary set in an uncountable regular cardinal can be σ-irreducible. (c) The argument in Theorem (2.2) would be vastly simplified if one knew that X had a σ-minimal base whose elements are convex sets. One cannot make such an assumption in the light of a result announced by Bennett and Berney in [BB_2], viz., if a GO space X has a σ-minimal base all of whose members are convex, then X is quasi-developable. The proof of that fact follows from [L_1, 5.11] once it is observed that any irreducible collection of convex sets is point-finite.

2.3 *Acknowledgement.* In the original version of this paper, and in the talk presented at the L.S.U. Conference, we asserted that any generalized ordered space having a σ-minimal base must be first countable. That is not true, as Dan Velleman pointed out to us. His example is a certain subset of the lexicographic product $\mathbb{Q} \times [0,\omega_1]$, where $\mathbb{Q}$ is the usual space of rational numbers; the relevant subspace is $X = \{(r,\alpha)\colon$ either α is a successor or $\alpha = \omega_1\}$. The authors are grateful to Dr. Velleman for pointing out this error.

3. Metrization of LOTS and σ-Minimal Bases

In this section we obtain structure theorems for certain LOTS having σ-minimal bases. Recall that a space X is *perfect* if every closed subset of X is a G_δ. We begin by presenting an example which shows the limits of the theory.

3.1 *Example. There is a perfect* LOTS *having a σ-minimal base which is not quasi-developable.*

Let D be a linearly ordered set such that (1) $|D| = c$;

(2) D has no end points; (3) ω_0 is cofinal in D and ω_0^*, the set ω_0 with the reversed ordering, is coinitial with D; (4) in its order topology, D is a discrete space. For example, if κ is the first ordinal with cardinality c, we could take D to be the lexicographically ordered set

$$D = [[0,\kappa) \times \omega_0^*] \cup [\{\kappa\} \times (\omega_0^* \cup \omega_0)].$$

Let P, Q and R be the sets of irrational, rational and real numbers respectively. Define X to be the set

$$X = (Q \times D) \cup (P \times \{0,1\})$$

with the lexicographic ordering.

Let $\mathcal{I}$ be the set of all open intervals in R having both end points rational. For each $J \in \mathcal{I}$ write $J =]r_J,s_J[$ and fix a 1-1 function $f_J: J \to D$. For each pair (q,J) where $q \in Q$, $J \in \mathcal{I}$ and $q < r_J$ and for each $x \in J \cap P$ define a set

$$B(x,q,J) = (Q \cap]r_J,x[) \times D \cup \langle (P \cap]r_J,x[) \times \{0,1\}\rangle \cup \cup \{(x,0)\} \cup \{(q,f_J(x))\}.$$

Then B(x,q,J) is an open subset of X. Define, for each $J \in \mathcal{I}$ and each $q \in Q$ having $q < r_J$, a collection $\mathcal{B}(q,J) = \{B(x,q,J) | x \in P \cap J\}$. We assert that each $\mathcal{B}(q,J)$ is irreducible. Fix $x \in P \cap J$. Then $(q,f_J(x)) \in B(x,q,J)$. And if $y \in P \cap J$ with $y \neq x$ then $f_J(x) \neq f_J(y)$ so that $(q,f_J(x)) \notin B(y,q,J)$, as required.

We assert that the collection $\mathcal{B} = \cup\{\mathcal{B}(q,J) | J \in I$ and $q \in Q$ has $q < r_J\}$ contains a base at each point (x,0) of X. For let $(x,0) \in U$, an open set in X. Then there is an irrational number $y < x$ such that $](y,0),(x,0)] \subset U$. Let J be any element of $\mathcal{I}$ having $y < r_J < x < s_J$. Choose any rational $q \in]y,r_J[$. Then $B(x,q,J) \in \mathcal{B}(q,J)$ and $B(x,q,J) \subset](y,0),(x,0)] \subset U$.

An analogous construction yields a σ-minimal collection of open sets which contains a base at each point $(x,1)$ of X. And finally, the collection $\mathcal{D} = \{\{(q,d)\} \mid q \in Q,\ d \in D\}$ is a minimal collection which contains a base at each point of $Q \times D$. Therefore X has a σ-minimal base.

Next X is perfect. For let V be any open set in X. Using countable cofinality and coinitiality of D, write $D = \bigcup_{n=1}^{\infty}[d_n,e_n]$ where $d_1 > d_2 > \cdots$ and $e_1 < e_2 < \cdots$. For each $q \in Q$ the set $\{q\} \times [d_n,e_n]$ is a closed discrete subset of X, so that each set $F(V,q,n) = V \cap \langle\{q\} \times [d_n,e_n]\rangle$ is a closed subset of X. Since $V \cap \langle Q \times D\rangle = \bigcup\{F(V,q,n) \mid q \in Q, n \geq 1\}$ we see that $V \cap \langle Q \times D\rangle$ is an F_σ in X. Now the subspace $P \times \{0,1\}$ of X is a part of the Alexandroff double arrow and is therefore perfect. Hence $V \cap \langle P \times \{0,1\}\rangle$ is an F_σ-subset of $\langle P \times \{0,1\}\rangle$, so that because $\langle P \times \{0,1\}\rangle$ is closed in X, $V \cap \langle P \times \{0,1\}\rangle$ is an F_σ in X. Then, being the union of two F_σ sets, V is an F_σ in X. Hence X is perfect.

Finally, X is not quasi-developable since $P \times \{0,1\}$ is a non-metrizable subspace of X.

We now turn to positive results concerning LOTS having σ-minimal bases.

3.2 *Proposition. Suppose* X *is a perfect* LOTS *having a σ-minimal base. Then* X *is a paracompact* p-*space.*

Proof. Let $\mathcal{B} = \bigcup\{\mathcal{B}(n) \mid n \geq 1\}$ be a σ-minimal base for X. For each $B \in \mathcal{B}(n)$ choose a point $p(B,n) \in B$ which belongs to no other member of $\mathcal{B}(n)$. Then the set $D(n) = \{p(B,n) \mid B \in \mathcal{B}(n)\}$ is a relatively closed, discrete subspace of the open set $\cup \mathcal{B}(n)$. Write $\cup \mathcal{B}(n) = \cup\{F(n,k) | k \geq 1\}$ where each $F(n,k)$ is closed in X, and let $D(n,k) = D(n) \cap F(n,k)$. Then $D = \cup\{D(n,k) | n,k \geq 1\}$ is a σ-discrete closed subspace of X so that, in the light of van Wouwe's theorem (cf. 1.3), X is a paracompact p-space.

3.3 *Remarks.* (a) The hypothesis in (3.2) that X is perfect is necessary. Consider the Michael line M [M] and the smallest LOTS M* containing M as a closed subspace [L_1, 2.7]. It is easily seen that M* is paracompact and quasi-developable, and therefore has a σ-minimal base hereditarily. Yet M* is *not* a p-space in the light of Theorem (1.2) since M is a non-metrizable subspace of M*.

(b) Of course it is true that a GO space which is perfect and has a σ-minimal base must also have a σ-discrete dense subspace. However, examples show that such spaces need not be p-spaces. For example, consider the space X of Example 3.1 and let $Y = X - (P \times \{1\})$. Then Y is a perfect GO space and Y has a σ-minimal base for the same reasons that X does. However Y cannot be a p-space because $P \times \{0\}$ is a closed subspace of Y which is not a p-space (and since any closed subspace Y of p-space must be a p-space).

Our next result is a corollary to certain work of van Wouwe [vW] or M. J. Faber [F]. However it is possible to give a self-contained proof and we provide it here.

3.4 *Proposition. Suppose* X *is a perfect, densely ordered*[2] LOTS *having a σ-minimal base. Then* X *is metrizable.*

[2]A linearly ordered set (X,<) is *densely ordered* if $]a,b[\neq \emptyset$ whenever $a < b$ in X. For example, any connected LOTS must be densely ordered.

Proof. Since X is perfect and collectionwise normal [S] by 1.5 it will be enough to show that X is quasi-developable. As in Proposition 3.2 find $E = \cup\{E(n) \mid n \geq 1\}$, a σ-discrete dense subspace of X. We may assume that $E(n) \subset E(n+1)$ for each $n \geq 1$ and that any end points of X are in E(1). Let $\mathcal{B} = \cup\{\mathcal{B}(n) \mid n \geq 1|$ be a σ-minimal base for X (not necessarily related in any way to the dense set E). Since E(n) is a closed discrete subset of the collectionwise normal space X there is a pairwise disjoint collection $\mathcal{V}(n) = \{V(x) \mid x \in E(n)\}$ of open sets such that $x \in V(x)$ for each $x \in E(n)$. Being perfect, X is first countable; for each $x \in E(n)$ let $\{V(x,n,k) \mid k \geq 1\}$ be a countable base of open neighborhoods for x with $V(x,n,k) \subset V(x)$ for every $x \in E(n)$. Let $\mathcal{E}(n,k) = \{V(x,n,k) \mid x \in E(n)\}$. Then $\cup\{\mathcal{E}(n,k) \mid n,k \geq 1\}$ is a σ-disjoint collection of open subsets of X which contains a neighborhood base at each point of E.

We now construct a σ-disjoint collection which acts as a base at points of X - E. For each $m,n \geq 1$ let $\mathcal{G}(m,n)$ be the collection of all convex components of the open set $(\cup\mathcal{B}(m)) - E(n)$. Suppose $p \in X - E$ and suppose $p \in U$ where U is open in X. Because $p \notin E$, p is not an end-point of X so that there are points $a, b \in X$ with $a < p < b$ and $]a,b[\subset U$. Since X is densely ordered, $]a,p[$ and $]p,b[$ are nonvoid open sets so that for a suitably large n we may choose points $r \in]a,p[\cap E(n)$ and $s \in]p,b[\cap E(n)$. Fix any m for which $p \in \cup \mathcal{B}(m)$. Let G be the convex component of $(\cup \mathcal{B}(m)) - E(n)$ which contains p. Then $G \subset]r,s[\subset U$ and $G \in \mathcal{G}(m,n)$. Since each $\mathcal{G}(m,n)$ is clearly pairwise disjoint we see that the collection $(\cup\{\mathcal{E}(n,k) \mid n;k \geq 1\}) \cup (\cup\{\mathcal{G}(m,n) \mid m,n \geq 1\})$ is a σ-disjoint base for X.

The hypothesis that X is densely ordered guarantees that there are no "jumps," i.e., pairs of consecutive points, in X. The above theorem is valid if the jumps are not too numerous, e.g., if the jumps constitute a σ-discrete subset of X. Example 3.1 shows the necessity of such an assumption.

4. Open Questions

The major open questions concerning ordered spaces having σ-minimal bases are:

> Suppose every closed subspace of a LOTS X has a σ-minimal base for its relative topology; must X be metrizable? Must X be metrizable if every subspace of X has a σ-minimal base for its topology? What if X is allowed to be a GO space instead of a LOTS?

The reader should be warned that one may *not* assume that X has a base $\mathcal{B}$ such that whenever $Y \subset X$ the collection $\{B \cap Y | B \in \mathcal{B}\}$ is σ-minimal; it is known that this additional hypothesis is sufficient to guarantee quasi-developability of X [BB_2]. There is one relatively easy consequence of the existence of σ-minimal bases for each closed subspace of a generalized ordered space X: such an X must be first countable. (Compare (2.3).)

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Texas Tech University

Lubbock, Texas 79409