TOPOLOGY PROCEEDINGS
Volume 2, 1977
Pages 401–407

http://topology.auburn.edu/tp/

# ON PERFECT SUBPARACOMPACTNESS AND A METRIZATION THEOREM FOR MOORE SPACES

by

J. Chaber and P. Zenor

**Topology Proceedings**

**Web:** http://topology.auburn.edu/tp/
**Mail:** Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
**E-mail:** topolog@auburn.edu
**ISSN:** 0146-4124

# ON PERFECT SUBPARACOMPACTNESS AND A METRIZATION THEOREM FOR MOORE SPACES

**J. Chaber and P. Zenor**

In this paper, we give a characterization of perfectly subparacompact spaces and we use this characterization to prove that perfectly normal subparacompact spaces which are locally connected and rim-compact are paracompact. In particular then, every perfectly normal, locally connected, and rim compact Moore space is metrizable. This generalizes the main theorem of [R,Z] and answers a question in [A,Z].

Recall that a space X is said to be subparacompact if for any open cover $\mathcal{U}$ of X there exists a sequence $\{\mathcal{V}(j)\}_{j\geq 1}$ of open covers of X such that if $x \in X$, then there is a $j \geq 1$ so that $St(x,\mathcal{V}(j)) \subseteq U$ for some $U \in \mathcal{U}$ (see [Bu]); X is perfectly subparacompact if X is subparacompact and closed subsets are $G_\delta$-sets in X. Also, X is rim-compact if for each point x of X and each open set V containing x, there is an open set W with a compact boundary so that $x \in W \subset V$.

We prove

*Theorem* 1. *Every perfectly normal, locally connected, rim-compact, and subparacompact space is paracompact.*

This yields

*Corollary. Every perfectly normal, locally connected and rim-compact Moore space is metrizable.*

The proof of Theorem 1 is based on the following

*Theorem* 2. *The following conditions are equivalent for a topological space* X:

*1)* X *is perfectly subparacompact,*

*2) for each well-ordered open cover* $\mathcal{U}$ *of* X *there exists a sequence* $\{\mathcal{V}(j)\}_{j\geq 1}$ *of open covers of* X *such that if* $x \in X$, *then there is a* $j \geq 1$ *such that* $\mathrm{St}(x,\mathcal{V}(j))$ *is contained in the first element of* $\mathcal{U}$ *that contains* x,

*3) for each well-ordered open cover* $\mathcal{U} = \{U_\alpha\}_{\alpha<\gamma}$ *of* X *there exists a sequence* $\{\mathcal{E}(j)\}_{j\geq 1}$ *of closed collections such that* $\mathcal{E}(j) = \{E_\alpha(j)\}_{\alpha<\gamma}$ *is increasing and* $\cup_{j\geq 1} E_\alpha(j) = \cup_{\beta<\alpha} U_\alpha$ *for* $\alpha < \gamma$,

*4) for each well-ordered open cover* $\mathcal{U} = \{U_\alpha\}_{\alpha<\gamma}$ *of* X *there exists a closed cover* $\mathcal{F} = \cup\mathcal{F}(j)$ *such that* $\mathcal{F}(j) = \{F_\alpha(j)\}_{\alpha<\gamma}$ *is discrete and* $F_\alpha(j) \subseteq U_\alpha \setminus \cup_{\beta<\alpha} U_\beta$ *for* $\alpha < \gamma$.

Theorem 2 was independently proved by H. Junnila.

In the first section, we shall prove Theorem 1 using the implication 1) $\to$ 4) of Theorem 2. The second section contains a proof of Theorem 2. We end by giving characterizations of perfectly paracompact spaces analogous to those given in Theorem 2.

## 1. Proof of Theorem 1

Let X be a space satisfying the hypothesis of Theorem 1 and let $\mathcal{U}$ be an open cover of X. We shall prove that $\mathcal{U}$ has a $\sigma$-discrete open refinement.

Since X is rim-compact, we may assume that all elements of $\mathcal{U}$ have compact boundaries. Furthermore, we may assume that $\mathcal{U} = \{U_\alpha\}_{\alpha<\gamma}$ is well-ordered.

From the condition 4) of Theorem 2 it follows that there exists a closed refinement $\mathcal{F} = \cup_{j\geq 1} \mathcal{F}(j)$ of $\mathcal{U}$ such that $\mathcal{F}(j) = \{F_\alpha(j)\}_{\alpha<\gamma}$ is discrete and $\mathcal{F}_\alpha(j) \subseteq U_\alpha \backslash \cup_{\beta<\alpha} U_\beta$.

Since $\mathcal{F}$ refines $\mathcal{U}$, it is sufficient to prove that each $\mathcal{F}(j)$ can be expanded to a $\sigma$-discrete open family. This follows from the second of the next two lemmas.

*Lemma* 1. *If a closed subset* F *of a regular rim-compact space* X *is contained in an open set* U, *then there is an open set* V *with a compact boundary such that* $F \subseteq V \subseteq \overline{V} \subseteq U$.

*Proof*. Let $\mathcal{W}$ be a finite open cover of the boundary of $\mathcal{U}$ consisting of sets with compact boundaries and such that $F \cap \overline{\cup\mathcal{W}} = \emptyset$. The set $V = U \backslash \overline{\cup\mathcal{W}}$ has the desired properties.

*Lemma* 2. *Let* X *be a perfectly normal locally connected and rim-compact space. If* $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha<\gamma}$ *is an open cover of* X, $\mathcal{F} = \{F_\alpha\}_{\alpha<\gamma}$ *is a discrete collection of closed subsets of* X *and, for* $\alpha < \gamma$

*(i)* $F_\alpha \subseteq U_\alpha$ *and the boundary of* $U_\alpha$ *is compact,*

*(ii)* $U_\alpha \cap (\cup_{\beta>\alpha} F_\beta) = \emptyset$,

*then there exists an open collection* $\mathcal{W} = \{W_\alpha\}_{\alpha<\gamma}$ *such that* $F_\alpha \subseteq W_\alpha$ *and* $\mathcal{W}$ *is a countable union of discrete collections.*

*Proof*. Since X is perfectly normal, we may choose a sequence $\{G_n\}_{n\geq 1}$ of open sets such that

(iii) $\cup\mathcal{F} = \cap_{n=1}^{\infty} G_n$ and $\overline{G}_{n+1} \subseteq G_n$ for $n \geq 1$.

Let $W_{\alpha,n}$ be the union of all components of $G_n$ intersecting $F_\alpha$. Since X is locally connected, each $W_{\alpha,n}$ is open. Moreover, from the definition of $W_{\alpha,n}$ it follows that for $\alpha_1 < \alpha < \gamma$

(iv) $W_{\alpha_1,n} \cap W_{\alpha,n} = \emptyset$ implies $W_{\alpha_1,n} \cap F_\alpha \neq \emptyset$.

We shall show that for each $\alpha < \gamma$ there exists an integer n such that

(v) $W_{\alpha,n} \cap [\cup_{\beta>\alpha} F_\beta] = \emptyset$.

Assume that this is not the case and take the smallest $\alpha$ such that $W_{\alpha,n} \cap [\cup_{\beta>\alpha} F_\beta] \neq \emptyset$ for all n.

By (i) and Lemma 1 there exists an open set V with a compact boundary such that $F_\alpha \subseteq V \subseteq \overline{V} \subseteq U_\alpha$. From (ii) it follows that the boundary C of V separates $F_\alpha$ from $\cup_{\beta>\alpha} F_\beta$. Since each $W_{\alpha,n}$ intersects $\cup_{\beta>\alpha} F_\beta$ and is the union of a collection of connected sets intersecting $F_\alpha$, the sets $W_{\alpha,n} \cap C$ are non-void. From the compactness of C and the fact that $G_{n+1} \subseteq G_n$, we have $A = \cap_{n\geq 1} \overline{W}_{\alpha,n} \cap C \neq \emptyset$. By (iii) and $W_{\alpha,n} \subseteq G_n$, $A \subseteq [\cup \mathcal{F}] \cap C$ but $C \cap [\cup_{\beta\geq\alpha} F_\beta] = \emptyset$. Hence there exists an $\alpha_1 < \alpha$ and an $x \in F_{\alpha_1} \cap \cap_{n\geq 1} \overline{W}_{\alpha,n}$.

Each $W_{\alpha,n}$ is closed in $G_n$ and $x \in G_n$, therefore $x \in \overline{W}_{\alpha,n}$ implies $x \in W_{\alpha,n}$ and (iv) shows that $\alpha_1$ does not satisfy (v) for any $n \geq 1$. This contradicts our choice of $\alpha$.

For each $\alpha < \gamma$, let $n(\alpha)$ be the first integer satisfying (v).

From (iv), the family $\mathcal{W}'_n = \{W_{\alpha,n}: n(\alpha) = n\}$ is pairwise disjoint. Since each $W_{\alpha,n}$ is a sum of components of $G_n$, $\mathcal{W}'_n$ is also discrete in $G_n$ and, consequently, $\mathcal{W}_n = \{G_{n+1} \cap W: W \in \mathcal{W}'_n\}$ is discrete in X. Thus $\mathcal{W} = \cup_{n\geq 1} \mathcal{W}_n$ is a $\sigma$-discrete open expansion of $\mathcal{F}$.

## 2. Proof of Theorem 2

It is easy to observe that the conditions 2), 3), and 4) are equivalent and imply 1). We shall prove 1) $\Rightarrow$ 2). In the proof of Theorem 1 we use 1) $\to$ 4). The proof of 2) $\Rightarrow$ 4) is a well known reasoning from [Bi]. H. Junnila gave a direct proof of 1) $\Rightarrow$ 4).

*Proof of* 1) $\Rightarrow$ 2). Let X be a perfectly subparacompact space and let $\mathcal{U} = \{U_\alpha\}_{\alpha<\gamma}$ be a well-ordered open cover of X. Since X is perfect, we can find, for each $\alpha < \gamma$, a sequence $\{E_\alpha(j)\}_{j\in N}$ of closed sets such that

(i) $\bigcup_{j\in N} E_\alpha(j) = \bigcup_{\beta<\alpha} U_\beta$

(Note, that in view of 3), we have to modify sets $E_\alpha(j)$ so that, for each $\alpha < \gamma$, $\{E_\alpha(j)\}_{j\in N}$ forma an increasing collection.)

For each $m \geq 0$ let $N^m$ denote the collection of all sequences of natural numbers of length m. If $t \in N^m$ and $i \in N$, then $(t,i) \in N^{m+1}$ denotes an extension of t by i. Put $T_u = \bigcup_{n\geq 0} N^{2n}$ and $T_v = \bigcup_{n\geq 0} N^{2n+1}$.

We will define, by induction on m, collections $\{\mathcal{U}(t)\}_{t\in T_u}$ and $\{\mathcal{V}(t)\}_{t\in T_v}$ of open covers of X such that $\mathcal{U}(\emptyset) = \mathcal{U}$ ($\emptyset \in T_u$ is the only element of $N^0$) and

(ii) if $t \in T_u$, then $\{\mathcal{V}(t,k)\}_{k\in N}$ is a sequence of open covers of X such that if $x \in X$, then there is a $k \in N$ such that $St(x,\mathcal{V}(t,k))$ is a subset of some element of $\mathcal{U}(t)$,

(iii) if $t \in T_v$ and $j \in N$, then $\mathcal{U}(t,j) = \{U_\alpha(t,j)\}_{\alpha<\gamma}$ where

$$U_\alpha(t,j) = U_\alpha \setminus (E_\alpha(j) \cup \overline{A}_\alpha(t)) \text{ and}$$
$$A_\alpha(t) = \{x \in X: St(x,\mathcal{V}(t)) \subseteq \bigcup_{\beta<\alpha} U_\beta\}.$$

If $t \in T_v$ and $\mathcal{V}(t)$ is an open cover of X, then the closure of the set $A_\alpha(t)$ defined in (iii) is contained in $\bigcup_{\beta<\alpha} U_\beta$, for all $\alpha < \gamma$. This, together with (i), shows that

$\mathcal{U}(t,j)$ defined in (iii) covers X for all $j \in N$.

If $t \in T_u$, then the collection $\{\mathcal{V}(t,k)\}_{k\in N}$ satisfying (ii) can be found by subparacompactness of X.

Therefore, starting with $\mathcal{U}(\emptyset) = \mathcal{U}$, we can define the collections $\{\mathcal{U}(t)\}_{t\in T_u}$ and $\{\mathcal{V}(t)\}_{t\in T_v}$ of open covers of X satisfying (ii) and (iii).

From the fact that $T_v$ is countable, it follows that it is sufficient to prove that if $x \in X$, then there exists a $t \in T_v$ such that $St(x,\mathcal{V}(t))$ is a subset of the first element of $\mathcal{U}$ that contains x.

Suppose that a point $x \in X$ does not have the above property and let $\alpha_x < \gamma$ and $t \in T_v$ be such that $St(x,\mathcal{V}(t)) \subseteq U_{\alpha_x}$ and $St(x,\mathcal{V}(t)) \subseteq U_\alpha$ implies $\alpha \geq \alpha_x$.

By our assumption $x \in \cup_{\beta<\alpha_x} U_\beta$ and, by (i), there is a $j \in N$ such that $x \in E_{\alpha_x}(j)$.

Consider $\mathcal{U}(t,j)$. From the definition of the sets $U_\alpha(t,j)$, it follows that $x \notin U_{\alpha_x}(t,j)$ (by $x \in E_{\alpha_x}(j)$) and $x \notin U_\alpha(t,j)$ for $\alpha > \alpha_x$ (by $St(x,\mathcal{V}(t)) \subseteq U_{\alpha_x}$). Therefore $x \notin \cup_{\alpha\geq\alpha_x} U_\alpha(t,j)$.

Since $(t,j) \in T_u$, by (ii), there exists a k and a $\beta < \gamma$ such that $St(x,\mathcal{V}(t,j,k)) \subseteq U_\beta(t,j)$. From $x \notin \cup_{\alpha\geq\alpha_x} U_\alpha(t,j)$, we have $\beta < \alpha_x$, but $U_\beta(t,j)$ is a subset of $U_\beta$ and this contradicts our choice of $\alpha_x$.

## 3. Theorem 3

*The following conditions are equivalent for a $T_1$ space X:*

*1) X is perfectly paracompact,*

*2) for each well-ordered open cover $\mathcal{U}$ of X there exists* *a sequence* $\{\mathcal{V}(j)\}_{j\geq 1}$ *of open covers of* X *such that if* $x \in X$, *then there exist a neighborhood* $O_x$ *of* x *and a* $j \geq 1$ *such that* $St(O_x, \mathcal{V}(j))$ *is contained in the first element of* $\mathcal{U}$ *that contains* x *(see* [A]*).*

*3) for each well-ordered open cover* $\mathcal{U} = \{U_\alpha\}_{\alpha<\gamma}$ *of* X *there exists a sequence* $\{\mathcal{E}(j)\}_{j\geq 1}$ *of closed collections such that* $\mathcal{E}(j) = \{E_\alpha(j)\}_{\alpha<\gamma}$ *is increasing and* $\cup_{j\geq 1} \text{Int } E_\alpha(j) = \cup_{j\geq 1} E_\alpha(j) = \cup_{\beta<\alpha} U_\beta$ *for* $\alpha < \gamma$.

Theorem 3 can be proved in the same way as Theorem 2. The implication 1) $\Rightarrow$ 2)(3)) can be also obtained as a corollary to the implication 1) $\Rightarrow$ 4) from Theorem 2 with the use of collectionwise normality of X.

## References

[AZ] K. Alster and P. L. Zenor, *On the collectionwise normality of generalized manifolds*, Top. Proc. 1 (1976), 125-127.

[A] A. Arhangel'skiǐ, *New criteria for paracompactness and metrizability of an arbitrary* $T_1$*-space*, Soviet Math. Dokl. 2 (1961), 1367-1369.

[Bi] R. H. Bing, *Metrization of topological spaces*, Canad. Journ. of Math. 3 (1951), 175-186.

[Bu] D. K. Burke, *On subparacompact spaces*, Proc. Amer. Math. Soc. 23 (1969), 655-663.

[RZ] G. M. Reed and P. L. Zenor, *Metrization of Moore spaces and generalized manifolds*, Fund. Math. XCl (1976), 203-210.

Auburn University

Auburn, AL 36830