TOPOLOGY PROCEEDINGS

Volume 2, 1977

Pages 431–444

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN: 0146-4124

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0. Introduction

For every nondegenerate Peano continuum X, the hyperspace 2^X of nonempty closed subsets of X, topologized by the Hausdorff metric, is homeomorphic to the Hilbert cube Q [4]. And for every proper nonempty closed subset A of X, the sub-hyperspaces $2^X(A) = \{F \in 2^X: F \cap A \neq \emptyset\}$ and $2^X_A = \{F \in 2^X: F \supset A\}$ are also homeomorphic to Q [5]. A natural question arises: how are $2^X(A)$ and 2^X_A situated in 2^X? In this paper we present some results on the topology of the hyperspace pairs $(2^X, 2^X(A))$ and $(2^X, 2^X_A)$.

The first general result in this direction was obtained in [3]: $2^X(A)$ is a Z-set in 2^X if and only if A is *locally non-separating in* X (i.e., for every nonempty connected open subset U of X, $U \setminus A$ is nonempty and connected). Of course, the condition that $2^X(A)$ be a Z-set in 2^X is equivalent to the existence of a pair homeomorphism between $(2^X, 2^X(A))$ and $([0,1] \times Q, \{0\} \times Q)$.

Thus for example, with $I = [0,1]$, the hyperspace $2^I(A)$ is a Z-set in 2^I if and only if $A \subset \partial I$. What then can be said about the topological position in 2^I of a hyperspace such as $2^I(\{1/2\}) = 2^I_{\{1/2\}}$? Since $2^I \setminus 2^I_{\{1/2\}}$ is the disjoint union $M_1 \cup M_2 \cup M_3$ of three copies of $Q \times [0,1)$, where $M_1 = \{F \in 2^I: F \subset [0,1/2)\}$, $M_2 = \{F \in 2^I: F \subset (1/2,1]\}$, and $M_3 = \{F \in 2^I: F \subset [0,1/2) \cup (1/2,1]$ and $F \cap [0,1/2) \neq \emptyset \neq F \cap (1/2,1]\}$, a first guess might be that $(2^I, 2^I_{\{1/2\}})$ is homeomorphic as a pair to $(T \times Q, \{v\} \times Q)$, where T is a union of three arcs meeting at a common endpoint v. On closer examination this is seen not to be the case (the closures of M_1 and M_2 do not contain $2^I_{\{1/2\}}$, and $2^I_{\{1/2\}}$ is not a Z-set in the closure of M_3). As it turns out, $(2^I, 2^I_{\{1/2\}})$ is not homeomorphic to a triangulated pair $(K \times Q, L \times Q)$ for *any* polyhedral pair (K,L). On the other hand, the hyperspace pair $(2^I, 2^I([1/3,2/3]))$ *is* triangulable. Let L be a 2-cell and $K = L \cup \alpha_1 \cup \alpha_2 \cup \alpha_3$, where the α_i are disjoint arcs attached to L such that each $\alpha_i \cap L$ is a boundary point of α_i and L. Then it is not difficult to show that $(2^I, 2^I([1/3,2/3]))$ is homeomorphic to $(K \times Q, L \times Q)$. These two examples motivated Theorem 1 below, which says that the hyperspace pair $(2^X, 2^X(A))$ is triangulable in the above sense if and only if A is locally non-separating in X at each point in bd A.

For those pairs $(2^X, 2^X(A))$ which are triangulable, Theorem 2 provides a simple topological classification relating to the components of X\cl(int A).

For the hyperspace pairs $(2^X, 2^X_A)$, Theorem 3 says that 2^X_A is a Z-set in 2^X if and only if A is not a finite subset of local cut points in X. Thus for example, 2^I_A is a Z-set in 2^I unless A is a finite subset of int I. In the latter case it can be shown that $(2^I, 2^I_A)$ is not even triangulable. It appears likely that in general, $(2^X, 2^X_A)$ is triangulable only if 2^X_A is a Z-set in 2^X.

1. Characterization of Triangulable Pairs $(2^X, 2^X(A))$

Definition. A Q-manifold pair (Y,M) is triangulable if there exists a polyhedral pair (K,L) such that $(Y,M) \approx (K \times Q, L \times Q)$.

Lemma 1. *If* $(2^X, 2^X(A))$ *is triangulable, then for every neighborhood* U *in* X *of a boundary point* p *of* A, *there exists a neighborhood* V *of* p *such that* V *meets only finitely many components of* $U \backslash A$.

Proof. Suppose not. Then there exists a neighborhood U of a point $p \in \mathrm{bd}\, A$, and a sequence $\{p_n\}$ in $U \backslash A$ converging to p, such that each p_n is in a distinct component of $U \backslash A$. It follows that for every neighborhood $\mathcal{U}$ in 2^X of $\{p\}$ such that $\cup\, \mathcal{U} \subset U$ (i.e., for every sufficiently small neighborhood $\mathcal{U}$ of $\{p\}$), the complement $\mathcal{U} \backslash 2^X(A)$ has infinitely many components. But this is impossible if $(2^X, 2^X(A)) \approx (K \times Q, L \times Q)$ for some polyhedral pair (K,L), since each point of $L \times Q$ has a basis of neighborhoods $\mathcal{W}$ in $K \times Q$ such that each complement $\mathcal{W} \backslash L \times Q$ has only finitely many components.

Definition. A closed subset A of a Peano continuum X is *locally non-separating in* X *at* $p \in \mathrm{bd}\, A$ if there exists at p a neighborhood base $\mathcal{U}(p)$ such that for each $U \in \mathcal{U}(p)$, $U \backslash A$ is connected.

It is routine to show that A is locally non-separating in X (as defined in §0) if and only if A has empty interior and is locally non-separating in X at each point.

Lemma 2. *Let* X *be a Peano continuum and* A *a closed subset which is locally non-separating in* X *at each boundary point. Then:*

i) $X \backslash cl(int\ A)$ *has a finite number of components* $\{G_i\}$,

ii) The closures $\{\overline{G}_i\}$ *are pairwise disjoint,*

iii) each $\overline{G}_i$ *is locally connected,*

iv) each $\overline{G}_i \cap A$ *is locally non-separating in* $\overline{G}_i$.

Proof. True if int $A = \emptyset$. Suppose int $A \neq \emptyset$. For each point $x \in$ bd(int A) there is an open neighborhood U in X such that $U \backslash A$ is connected. Since $U \backslash A$ is dense in $U \backslash$cl(int A), it follows that $U \backslash$cl(int A) is connected. By compactness, bd(int A) is covered by a finite collection $\{U_i\}$ of such neighborhoods. Since X is connected, each component of $X \backslash$cl(int A) has a limit point in bd(int A). Then each component must intersect, and therefore contain, some $U_i \backslash$cl(int A). Thus the number of components of $X \backslash$cl(int A) is finite. And since $\overline{G}_i \backslash G_i \subset$ bd(int A), it follows also that $\overline{G}_i \cap \overline{G}_j = \emptyset$ if $i \neq j$.

Consider $x \in$ bd $G_i = \overline{G}_i \cap$ cl(int A). As shown above, x has arbitrarily small neighborhoods U in X such that $U \backslash$cl(int A) is connected. Since $U \backslash$cl(int A) is dense in $U \cap \overline{G}_i$, the latter set is a connected neighborhood of x in $\overline{G}_i$. Thus $\overline{G}_i$ is locally connected.

Clearly, $\overline{G}_i \cap A$ is nowhere dense in $\overline{G}_i$, and A is locally non-separating in $\overline{G}_i$ at each point of $A \cap G_i$. For $x \in$ bd G_i the basic neighborhoods $U \cap \overline{G}_i$ of x obtained above are such that $U \cap \overline{G}_i \backslash A = U \backslash A$ is connected. Thus $\overline{G}_i \cap A$ is locally non-separating in $\overline{G}_i$.

Definition. A *strong* Q-*decomposition of a pair* (Y,M) is a finite cover $\{Y_i\}$ of Y such that:

i) each decomposition element Y_i is homeomorphic to Q,

ii) each nonempty intersection $Y_i \cap Y_j$ is a union of decomposition elements,

iii) Y_i is a Z-set in Y_j whenever $Y_i \subsetneq Y_j$,

iv) M is a union of decomposition elements.

Lemma 3. *A compact* Q-*manifold pair* (Y,M) *is triangulable if and only if it admits a strong* Q-*decomposition.*

Proof. This is similar to the proof of Theorem 2.4 of [2]. Let $\{Y_i\}$ be a strong Q-decomposition of the pair (Y,M). We construct a simplicial complex K which is the union of a collection $\{K_i\}$ of subcomplexes in 1-1 correspondence with the decomposition elements $\{Y_i\}$, and a homeomorphism $h: Y \to K \times Q$ such that for each i, $h(Y_i) = K_i \times Q$. Then for $L = \cup\{K_i: Y_i \subset M\}$, we have $(Y,M) \approx (K \times Q, L \times Q)$. The construction is inductive, beginning with the minimal elements of the decomposition.

To this end, we write $\{Y_i\}$ as a monotone union of subcollections, $\emptyset = Y^{(-1)} \subset Y^{(0)} \subset \cdots \subset Y^{(m)} = \{Y_i\}$, such that if $Y_j \subsetneq Y_k \in Y^{(n)}$, then $Y_j \in Y^{(n-1)}$. Thus each element of $Y^{(0)}$ is a minimal decomposition element. Let $K^{(0)}$ be a collection of points in 1-1 correspondence with the elements of $Y^{(0)}$, and choose a corresponding homeomorphism $h^{(0)}$: $\cup Y^{(0)} \to K^{(0)} \times Q$. Inductively, suppose there exists a complex $K^{(n-1)}$ and a homeomorphism $h^{(n-1)}: \cup Y^{(n-1)} \to K^{(n-1)} \times Q$ such that for each $Y_j \in Y^{(n-1)}$, $h^{(n-1)}(Y_j) = K_j \times Q$ for some subcomplex K_j of $K^{(n-1)}$. Then for each $Y_i \in Y^{(n)} \setminus Y^{(n-1)}$, set $K_i = \text{cone } (\cup\{K_j: Y_j \subsetneq Y_i\})$. The cone points are chosen so that each $K_i \cap K^{(n-1)} = \cup\{K_j: Y_j \subsetneq Y_i\}$, and $K_{i_1} \cap K_{i_2} \subset K^{(n-1)}$ for each $Y_{i_1}, Y_{i_2} \in Y^{(n)} \setminus Y^{(n-1)}$. Now define

$K^{(n)} = \cup\{K_i : Y_i \in Y^{(n)}\setminus Y^{(n-1)}\} \cup K^{(n-1)}$, and construct the homeomorphism $h^{(n)} : \cup Y^{(n)} \to K^{(n)} \times Q$ by requiring the restriction of $h^{(n)}$ to $\cup Y^{(n-1)}$ to agree with $h^{(n-1)}$, and taking $h^{(n)}(Y_i) = K_i \times Q$ for each $Y_i \in Y^{(n)}\setminus Y^{(n-1)}$. (The latter is accomplished by simply applying the Z-set homeomorphism extension theorem to each such Y_i and $K_i \times Q$). This completes the inductive step. The complex $K = K^{(m)}$ and the homeomorphism $h = h^{(m)} : Y \to K \times Q$ fulfill the requirements.

For the converse, it is obvious that if $(Y,M) \approx (K \times Q, L \times Q)$, it admits a strong Q-decomposition.

For closed subsets $A_1,\cdots,A_n$ of a nondegenerate Peano continuum X, let $2^X(A_1,\cdots,A_n) = \{F \in 2^X : F \cap A_i \neq \emptyset$ for each $i\}$. It was shown in [5] that $2^X(A_1,\cdots,A_n) \approx Q$. We also require the following result from [3].

Lemma 4. *For* $B \in 2^X$, *the hyperspace* $2^X(A_1,\cdots,A_n,B)$ *is a Z-set in* $2^X(A_1,\cdots,A_n)$ *if and only if* B *is locally non-separating in* X *and* $A_i\setminus B$ *is dense in* A_i, *for each* i.

Theorem 1. *The hyperspace pair* $(2^X,2^X(A))$ *is triangulable if and only if* A *is locally non-separating in* X *at each boundary point.*

Proof. Suppose first that $(2^X,2^X(A))$ is triangulable, and suppose that A is locally separating in X at some boundary point p. Then there exists a neighborhood U of p such that for each neighborhood $V \subset U$, $V\setminus A$ is separated.

Consider the element $\{p\}$ of $2^X(A)$. For any neighborhood $\mathcal{U}$ of $\{p\}$ in 2^X such that $\cup\,\mathcal{U} \subset U$, $\mathcal{U}\setminus 2^X(A)$ is separated, since there exist points in X arbitrarily close to p which lie in different components of $U\backslash A$. Thus $2^X(A)$ is locally separating in 2^X at $\{p\}$.

We show next that there exist elements of $2^X(A)$ arbitrarily close to $\{p\}$ at which $2^X(A)$ is locally non-separating in 2^X. By Lemma 1 there exists a monotone decreasing sequence $\{V_n\}$ of neighborhoods of p such that $V_1 \subset U$, diam $V_n \to 0$, and V_{n+1} meets only finitely many components of $V_n\backslash A$, for each n. Choose a finite subset F_n of $V_n\backslash A$, consisting of one point from each component of $V_n\backslash A$ meeting V_{n+1}. It may be assumed that $V_{n+2} \cap F_n = \emptyset$. We claim that $2^X(A)$ is locally non-separating in 2^X at elements of the form $\cup\{F_{2n}: n \geq N\} \cup \{p\}$, for each N. Given $\varepsilon > 0$, choose $m > N$ such that diam $V_{2m} < \varepsilon$, and choose a connected neighborhood $\mathcal{N}$ in $2^X\backslash 2^X(A)$ of $\cup\{F_{2n}: N \leq n < m\}$ such that diam $\mathcal{N} < \varepsilon$ and $(\cup\mathcal{N}) \cap V_{2m} = \emptyset$. Define a neighborhood $\mathcal{V}$ of $\cup\{F_{2n}: n \geq m\} \cup \{p\}$ as follows. Let $\{G_i\}$ be the finite collection of components of $V_{2m}\backslash A$ meeting V_{2m+1}. Then take $\mathcal{V} = \{B \in 2^X: B \subset \cup\{G_i\} \cup V_{2m+1}$ and $B \cap G_i \neq \emptyset$ for each i$\}$. We have diam $\mathcal{V} < \varepsilon$, and $\mathcal{N} \times \mathcal{V} = \{B_1 \cup B_2: B_1 \in \mathcal{N}$ and $B_2 \in \mathcal{V}\} \subset 2^X$ is a neighborhood of $\cup\{F_{2n}: n \geq N\} \cup \{p\}$ with diameter less than ε. Clearly, $\mathcal{N} \times \mathcal{V}\backslash 2^X(A) = \mathcal{N} \times (\mathcal{V}\backslash 2^X(A)) = \mathcal{N} \times \Pi\{2^G: G \in \{G_i\}\}$ is connected.

Of course, the above element $\cup\{F_{2n}: n \geq N\} \cup \{p\}$ of $2^X(A)$ is arbitrarily close to elements $\cup\{F_{2n}: N \leq n \leq M\} \cup \{p\}$ of $2^X(A)$, at which $2^X(A)$ is locally separating in 2^X. In turn, these latter elements are arbitrarily close to elements $\cup\{F_{2n}: N \leq n \leq M\} \cup \cup \{F_{2n}: n \geq N_1\} \cup \{p\}$ of $2^X(A)$ at which $2^X(A)$ is locally non-separating in 2^X.

Thus we may inductively choose a sequence $\{S_i\}$ in $2^X(A)$ with $S_1 = \{p\}$, each S_{i+1} arbitrarily close to S_i, and with the following alternating property: $2^X(A)$ is locally separating in 2^X at $S_1, S_3, \cdots$ and locally non-separating at $S_2, S_4, \cdots$. By hypothesis there exists a homeomorphism $h: (2^X, 2^X(A)) \to (K \times Q, L \times Q)$, for some polyhedral pair (K,L). Then $h(S_1) \in \text{int } \sigma_1 \times Q$, for some simplex σ_1 of L. Since $L \times Q$ is locally separating in $K \times Q$ at $h(S_1)$, it follows that $L \times Q$ is locally separating in $K \times Q$ at each point of $\text{int } \sigma_1 \times Q$. Therefore, if S_2 is close enough to S_1, $h(S_2) \in \text{int } \sigma_2 \times Q$ for some simplex σ_2 of L properly containing σ_1. Then $L \times Q$ is locally non-separating in $K \times Q$ at each point of $\text{int } \sigma_2 \times Q$, and if S_3 is close enough to S_2 we must have $h(S_3) \in \text{int } \sigma_3 \times Q$ for some simplex σ_3 properly containing σ_2. Continuing, we obtain an infinite ascending sequence $\sigma_1 \subsetneq \sigma_2 \subsetneq \sigma_3 \cdots$ of simplices of L, contradicting the compactness of L. Thus if $(2^X, 2^X(A))$ is triangulable, then A must be locally non-separating in X at each boundary point.

Conversely, suppose that A is locally non-separating in X at each boundary point. If A is nowhere dense then $2^X(A)$ is a Z-set in 2^X, and $(2^X, 2^X(A)) \approx ([0,1] \times Q, \{0\} \times Q)$. Now suppose that $\text{int } A \neq \emptyset$; using Lemma 2, let $\mathcal{C} = \{G_i\}$ be the finite collection of components of $X \setminus \text{cl}(\text{int } A)$. Note that $\text{bd } G_i = \overline{G_i} \cap \text{bd}(\text{int } A)$ for each i. We partition $\mathcal{C}$ into four subcollections as follows:

1) $\mathcal{C}_1 = \{G \in \mathcal{C}: G \cap A = \emptyset\}$,
2) $\mathcal{C}_2 = \{G \in \mathcal{C}: G \cap A \neq \emptyset$ but $\overline{G \cap A} \cap \text{bd } G$ is nowhere dense in $\text{bd } G\}$,
3) $\mathcal{C}_3 = \{G \in \mathcal{C}: \overline{G \cap A} \cap \text{bd } G$ has nonempty interior in $\text{bd } G$ but $\overline{G \cap A} \neq \text{bd } G\}$,

4) $\mathcal{C}_4 = \{G \in \mathcal{C}: \overline{G \cap A} \supset \text{bd } G\}$.

For each $G \in \mathcal{C}$ we consider a collection $\mathcal{H}(G)$ of subspaces of $2^{\overline{G}}$ as follows:

i) $\mathcal{H}(G) = \{2^{\overline{G}}, 2^{\overline{G}}(\text{bd } G)\}$ if $G \in \mathcal{C}_1$,

ii) $\mathcal{H}(G) = \{2^{\overline{G}}, 2^{\overline{G}}(\text{bd } G), 2^{\overline{G}}(\overline{G \cap A}), 2^{\overline{G}}(\text{bd } G, \overline{G \cap A}\}$ if $G \in \mathcal{C}_2$,

iii) $\mathcal{H}(G) = \{2^{\overline{G}}, 2^{\overline{G}}(\overline{G \cap A}), 2^{\overline{G}}(\text{bd } G \cap \overline{G \cap A}),$
$2^{\overline{G}}(\text{cl}(\text{bd } G \backslash \overline{G \cap A})), 2^{\overline{G}}(\text{bd } G \cap \overline{G \cap A}, \text{cl}(\text{bd } G \backslash \overline{G \cap A})),$
$2^{\overline{G}}(\overline{G \cap A}, \text{cl}(\text{bd } G \backslash \overline{G \cap A}))\}$ if $G \in \mathcal{C}_3$,

iv) $\mathcal{H}(G) = \{2^{\overline{G}}, 2^{\overline{G}}(\text{bd } G), 2^{\overline{G}}(\overline{G \cap A})\}$ if $G \in \mathcal{C}_4$.

For each nonempty subcollection $\mathcal{D} = \{G_{i_1}, \cdots, G_{i_k}\}$ of $\mathcal{C}$, define $\mathcal{H}(\mathcal{D}) = \{\Pi_{j=1}^k \mathcal{H}_{i_j}: \mathcal{H}_{i_j} \in \mathcal{H}(G_{i_j})\}$, where $\Pi_{j=1}^k \mathcal{H}_{i_j} = \{F \in 2^X: F = \cup_{j=1}^k F_{i_j}$ with each $F_{i_j} \in \mathcal{H}_{i_j}\}$. We claim that the collection $\cup\{\mathcal{H}(\mathcal{D}): \emptyset \neq \mathcal{D} \subset \mathcal{C}\} \cup \{2^X(\text{cl}(\text{int } A))\}$ of subspaces of 2^X is a strong Q-decomposition of the pair $(2^X, 2^X(A))$. The verification is routine. In particular, the necessary Z-set conditions are in most instances consequences of Lemma 2 and Lemma 4. The only exceptions are in situations like $2^{\overline{G}}(\text{bd } G) \subsetneq 2^X(\text{cl}(\text{int } A))$, where a "fattening" of hyperspace elements via a convex metric on X provides a small push of $2^X(\text{cl}(\text{int } A))$ into $2^X(\text{cl}(\text{int } A)) \backslash 2^{\overline{G}}(\text{bd } G)$ (see the proof of Lemma 4.2 of [5]). We conclude by Lemma 3 that $(2^X, 2^X(A))$ is triangulable.

2. Classification of Triangulable Pairs $(2^X, 2^X(A))$

Suppose A is locally non-separating in X at each boundary point, and int $A \neq \emptyset$. We consider the partition $\cup_{i=1}^4 \mathcal{C}_i$ of the collection $\mathcal{C}$ of components of $X \backslash \text{cl}(\text{int } A)$, as described in the proof of Theorem 1. For each i, let

$\tau_i(X,A)$ be the cardinality of $\mathcal{C}_i$, and define the 4-tuple $\tau(X,A) = (\tau_i(X,A))_{i=1}^4$. It is easily seen that all possible values for $\tau(X,A)$ are realized (for example, take cl(int A) to be a 2-cell, with the closure $\overline{G}$ of each component of X\cl(int A) a 2-cell meeting cl(int A) along a common boundary arc). Note that $\tau(X,A) = (0,0,0,0)$ if and only if $A = X$.

Theorem 2. *Triangulable pairs* $(2^X,2^X(A))$ *and* $(2^Y,2^Y(B))$ *are homeomorphic if and only if either int* $A = \emptyset =$ *int* B *or* $\tau(X,A) = \tau(Y,B)$.

Proof. If int $A = \emptyset =$ int B, then $(2^X,2^X(A)) \approx ([0,1] \times Q,\{0\} \times Q) \approx (2^Y,2^Y(B))$. If $\tau(X,A) = \tau(Y,B)$, the strong Q-decompositions of the pairs $(2^X,2^X(A))$ and $(2^Y,2^Y(B))$ constructed in the proof of Theorem 1 are obviously isomorphic. Then the same polyhedral pair (K,L) is associated with each decomposition, and $(2^X,2^X(A)) \approx (K \times Q, L \times Q) \approx (2^Y,2^Y(B))$.

Conversely, suppose $(2^X,2^X(A)) \approx (2^Y,2^Y(B))$. Then either int $A = \emptyset =$ int B or int $A \neq \emptyset \neq$ int B. In the latter case we show that $\tau(X,A) = (\tau_1,\tau_2,\tau_3,\tau_4)$ is a topological invariant of $(2^X,2^X(A))$, hence $\tau(X,A) = \tau(Y,B)$. Clearly, the number of components of $2^X\backslash 2^X(A)$ is equal to the number of nonempty subcollections of $\mathcal{C}$. Thus $\tau_1 + \tau_2 + \tau_3 + \tau_4$ is a topological invariant of $(2^X,2^X(A))$. Since the number of components of $2^X\backslash 2^X(A)$ whose closures intersect $2^X(A)$ only in cl(int $2^X(A)$) = 2^X(cl(int A)) is equal to the number of nonempty subcollections of $\mathcal{C}_1, \tau_1$ is an invariant. Since the number of components $\mathcal{K}$ of $2^X\backslash 2^X(A)$ for which bd $\mathcal{K}\backslash$cl(int $2^X(A)$) is dense in bd $\mathcal{K}$ is equal to the number of nonempty subcollections of $\mathcal{C}_4$, τ_4 is an invariant. Finally, the number of components $\mathcal{K}$ of $2^X \backslash 2^X(A)$ for which $\mathrm{cl}(\mathrm{bd}\ \mathcal{K} \backslash \mathrm{cl}(\mathrm{int}\ 2^X(A))$ does not contain a nonempty relatively open set in $\mathrm{bd}\ \mathcal{K} \cap \mathrm{cl}(\mathrm{int}\ 2^X(A))$ is equal to the number of nonempty subcollections of $\mathcal{C}_1 \cup \mathcal{C}_2$. Thus $\tau_1 + \tau_2$ is an invariant. Then $\tau_2 = (\tau_1 + \tau_2) - \tau_1$ and $\tau_3 = (\tau_1 + \tau_2 + \tau_3 + \tau_4) - (\tau_1 + \tau_2) - \tau_4$ are also invariants, thus $\tau(X,A)$ is an invariant as claimed.

3. Characterization of Z-set Pairs $(2^X, 2^X_A)$

Theorem 3. *Let* X *be a nondegenerate Peano continuum and* A *a nonempty closed subset. Then* 2^X_A *is a* Z-*set in* 2^X *if and only if* A *is not a finite set of local cut points in* X.

Proof. If A contains a point p which is not a local cut point in X (i.e., $\{p\}$ is locally non-separating in X), then $2^X_{\{p\}} = 2^X(\{p\})$ is a Z-set in 2^X, thus $2^X_A \subset 2^X_{\{p\}}$ is also a Z-set in 2^X.

Now suppose A is an infinite set. We show that 2^X_A is a Z-set in 2^X by an argument adapted from the proof of Lemma 5.4 of [5] (in which it was shown that 2^X_A is a Z-set in 2^X whenever $\mathrm{int}\ A \neq \emptyset$). Given $\varepsilon > 0$, let $\mathcal{P}$ be a partition of X with mesh less than $\varepsilon/3$. That is, $\mathcal{P}$ is a finite disjoint collection of connected open subsets with diameters less than $\varepsilon/3$ and whose closures cover X. We may further suppose that the closure of each partition element is locally connected, and that some partition element α contains a cluster point of A. There exists a finite connected graph (in fact, a tree) T in the Peano continuum $\overline{\alpha}$ such that $M = \cup\{\overline{\beta}\colon\ \overline{\alpha} \cap \overline{\beta} \neq \emptyset, \alpha \neq \beta \in \mathcal{P}\} \cup T$ is connected, and therefore a Peano continuum. Then the hyperspace 2^M is an AR, and there exists a map

$r: \overline{\alpha} \Rightarrow 2^M$ such that $r(x) = \{x\}$ for each $x \in \text{bd}\, \alpha$. Extend r to a map $s: X \Rightarrow 2^X$ by setting $s(x) = \{x\}$ for each $x \in X \backslash \overline{\alpha}$. Note that $\rho(\{x\}, s(x)) < 2\varepsilon/3$ for all x. Define the map $f: 2^X \Rightarrow 2^X$ by $f(F) = \cup\{s(x): x \in F\}$. Then $\rho(f,\text{id}) < 2\varepsilon/3$, and $f(F) \cap \alpha \subset T$ for each $F \in 2^X$.

If $A \cap \alpha \not\subset T$, then $f(F) \not\supset A$ for each F, hence f maps into $2^X \backslash 2^X_A$, and 2^X_A is a Z-set in 2^X. On the other hand, if $A \cap \alpha \subset T$ then $A \cap T \cap \alpha$ is infinite (recall that α contains a cluster point of A), and there exists an arc J in $T \cap \alpha$ containing infinitely many points of A. We may assume that J is a free arc in the Peano continuum $(X \backslash \alpha) \cup T = \cup\{\overline{\beta}: \alpha \neq \beta \in \mathcal{P}\} \cup T$. Let a_1, a_2 be distinct points of $A \cap \text{int}\, J$. There is constructed in the proof of Lemma 5.4 of [5] a map $g: 2^{(X\backslash\alpha) \cup T} \Rightarrow 2^{(X\backslash\alpha) \cup T} \backslash 2^{(X\backslash\alpha) \cup T}_{\{a_1,a_2\}}$ such that $\rho(g,\text{id}) <$ diam J. Then the composition gf maps 2^X into $2^X \backslash 2^X_{\{a_1,a_2\}} \subset 2^X \backslash 2^X_A$, and since diam $J <$ diam $\alpha < \varepsilon/3$, $\rho(gf,\text{id}) < 2\varepsilon/3 + \varepsilon/3 = \varepsilon$. Thus 2^X_A is a Z-set in 2^X if A is infinite or if A contains a point which is not a local cut point in X.

Conversely, suppose $A = \{y_1, \cdots, y_n\}$ with each y_i a local cut point in X. We show that for each sufficiently small neighborhood $\mathcal{U}$ of the element A in 2^X, $\mathcal{U} \backslash 2^X_A$ is not $(n-1)$-connected. There exist disjoint connected open neighborhoods V_i of y_i in X, $i = 1, \cdots, n$, such that $V_i \backslash \{y_i\} = V_i^- \cup V_i^+$ is a separation. Let d be a convex metric on X. For each i, define a map $\pi_i: \{F \in 2^X: F \subset V_i\} \Rightarrow (-\infty, \infty)$ by

$$\pi_i(F) = \begin{cases} -d(y_i,F) & \text{if } F \subset V_i^-, \\ d(y_i,F) & \text{if } F \cap V_i^+ \neq \emptyset, \\ 0 & \text{if } y_i \in F. \end{cases}$$

With $\mathcal{V} = \{F \in 2^X: F \subset \cup_1^n V_i$ and $F \cap V_i \neq \emptyset$ for each $i\}$, a map $\pi: \mathcal{V} \Rightarrow \Pi_1^n(-\infty,\infty)$ is defined by $\pi(F) = (\pi_i(F \cap V_i))_1^n$. Note that $\pi^{-1}(0,\cdots,0) = \mathcal{V} \cap 2_A^X$. Since the closure of each component of $V_i \backslash \{y_i\}$ must contain y_i, there is for each i an arc α_i in V_i such that $x \Rightarrow \pi_i(\{x\})$ defines a homeomorphism of α_i onto some interval $[-t_i,t_i]$. Let $g: S^{n-1} \Rightarrow \Pi_1^n[-t_i,t_i] \backslash (0,\cdots,0)$ be any essential map, and let $\tilde{g}: S^{n-1} \Rightarrow \mathcal{V} \backslash 2_A^X$ be the lifting of g via the arcs $\{\alpha_i\}$. That is, $\tilde{g}(s) = \cup_1^n\{x_i\}$, where each $x_i \in \alpha_i$, and $\pi\tilde{g} = g$.

For any neighborhood $\mathcal{U}$ of A in 2^X such that $\mathcal{U} \subset \mathcal{V}$, we may ensure that $\tilde{g}$ maps into $\mathcal{U}$ by requiring that g map into a small neighborhood of $(0,\cdots,0)$. And clearly, the map $\tilde{g}: S^{n-1} \Rightarrow \mathcal{U} \backslash 2_A^X$ is not homotopic to a constant map, since composing such a homotopy with π would provide a homotopy from $g: S^{n-1} \Rightarrow \Pi_1^n(-\infty,\infty) \backslash (0,\cdots,0)$ to a constant map. Thus 2_A^X cannot be a Z-set in 2^X.

Conjecture. The pair $(2^X, 2_A^X)$ is triangulable only if 2_A^X is a Z-set in 2^X.

It can be shown, by a strategy similar to that employed in the proof of Theorem 1, that if $A = \{y_1,\cdots,y_n\}$ where each y_i is a local cut point of finite order, then $(2^X, 2_A^X)$ is not triangulable.

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