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ON THE CANCELLATION OF SHAPES FROM PRODUCTS

J. G. Hollingsworth and R. B. Sher

K. Borsuk [1] has raised the following question concerning the "cancellation law" for shapes of compacta: Is it true that $Sh(X)\times Sh(S^1) = Sh(Y)\times Sh(S^1)$ implies that $Sh(X) = Sh(Y)$? The answer to this question is generally "no." Specifically, Charlap [2] has constructed compact manifolds M and N such that $M \not\cong N$ and $M\times S^1 \cong N\times S^1$. (In fact, M and N are Riemannian manifolds and $M\times S^1$ is diffemorphic to $N\times S^1$.) As in the case of the analogous question in the homotopy theory of compact ANR's, the problem lies in the fundamental group. Thus, in this brief note we are able to establish the following.

Theorem. Suppose X *and* Y *are approximatively* 1-*connected compacta and* $Sh(X)\times Sh(S^1) = Sh(Y)\times Sh(S^1)$. *Then* $Sh(X) = Sh(Y)$.

For our purposes, we shall find most suitable the ANR-systems approach to shape due to Mardešić and Segal [3]. By [4], this approach may be regarded as equivalent to Borsuk's.

We shall adopt, along with the terminology of [3], the following notations.

(i) R shall denote the real numbers;

(ii) S^1 is the set of complex numbers z such that $|z| = 1$;

(iii) $\exp: R \to S^1$ is the exponential map;

(iv) If A is a space, then $\alpha: A \to A\times S^1$ is defined by $\alpha(a) = (a,1)$ for all $a \in A$;

(v) If A and B are spaces, $\pi: A\times B \to A$ is defined by $\pi(a,b) = a$ for all $(a,b) \in A\times B$.

Of course α and π depend on the space A and, in the case of π, B. However, context shall make the meaning clear in each case.

Proof of the theorem. We may regard X and Y as being subsets of Q, the Hilbert cube. Let $\underline{X} : U_1 \supset U_2 \supset U_3 \supset \cdots$ be an inclusion ANR-sequence for X in Q and $\underline{Y} : V_1 \supset V_2 \supset V_3 \supset \cdots$ be an inclusion ANR-sequence for Y in Q. Then $\underline{X}' : U_1\times S^1 \supset U_2\times S^1 \supset U_3\times S^1 \supset \cdots$ and $\underline{Y}' : V_1\times S^1 \supset V_2\times S^1 \supset V_3\times S^1 \supset \cdots$ are inclusion ANR-sequences for $X\times S^1$ and $Y\times S^1$, respectively, in $Q\times S^1$. Hence, there exist system maps $\underline{f}' = (f',f_i') : \underline{X}' \to \underline{Y}'$ and $\underline{g}' = (g',g_i') : \underline{Y}' \to \underline{X}'$ such that $\underline{g}'\underline{f}' \simeq \underline{1}_{X'}$ and $\underline{f}'\underline{g}' \simeq \underline{1}_{Y'}$. Let $f = f'$ and, for each positive integer i, $f_i = \pi f_i'\alpha : U_{f(i)} \to V_i$. Similarly, let $g \doteq g'$ and $g_i = \pi g_i'\alpha : V_{g(i)} \to U_i$. The proof will be complete if we are able to establish that:

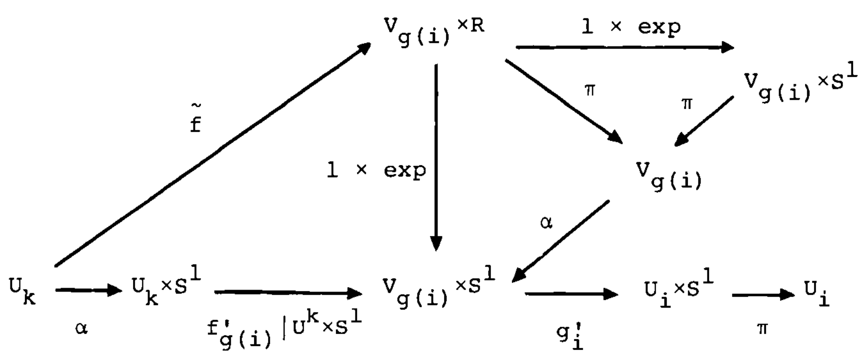
(1) $\underline{f} = (f,f_i) : \underline{X} \to \underline{Y}$ and $\underline{g} = (g,g_i) : \underline{Y} \to \underline{X}$ are system maps, and

(2) $\underline{g}\ \underline{f} \simeq \underline{1}_X$ and $\underline{f}\ \underline{g} \simeq \underline{1}_Y$.

Assertion (1) is easily verified, and we shall not present the details. The reader familiar with shape theory will note that there are obvious system maps $\underline{\alpha}: \underline{X} \to \underline{X}'$ and $\underline{\pi}: \underline{Y}' \to \underline{Y}$, associated with the maps $\alpha: X \to X\times S^1$ and $\pi: Y\times S^1 \to Y$ respectively, so that $\underline{f} = \underline{\pi}\ \underline{f}'\underline{\alpha}$. A similar comment applies to $\underline{g}$.

To confirm the first part of (2), we are required to show that if i is a positive integer, then there exists an integer $k \geq f(g(i))$ such that $1_{U_k} \simeq g_i f_{g(i)}|U_k$ in U_i. Since $\underline{g}'\underline{f}' \simeq \underline{1}_{\underline{X}'}$, we know that there exist a positive integer $j \geq f(g(i))$ and a homotopy $H : U_j \times S^1 \times I \to U_i \times S^1$ such that for all $(x,y) \in U_j \times S^1$, $H(x,y,0) = (x,y)$ and $H(x,y,1) = g_i^! f'_{g(i)}(x,y)$. Since X is approximatively 1-connected, there exists an integer $k \geq j$ such that each loop in U_k is contractible in U_j. Define $G : U_k \times I \to U_i$ by $G(x,t) = \pi H(x,1,t)$ for all $x \in U_k$ and $t \in I$. Then, if $x \in U_k$, $G(x,0) = \pi H(x,1,0) = \pi(x,1) = x$ and $G(x,1) = \pi H(x,1,1) = \pi g_i^! f'_{g(i)}(x,1) = \pi g_i^! f'_{g(i)}\alpha(x)$. We conclude that $1_{U_k} \simeq \pi g_i^! f'_{g(i)}\alpha$ in U_i.

Since each loop in U_k is contractible in U_j, the map $(f'_{g(i)}|U_k \times S^1)\alpha\colon U_k \to V_{g(i)} \times S^1$ can be lifted through the covering $1_{V_{g(i)}} \times \exp\colon V_{g(i)} \times R \to V_{g(i)} \times S^1$ to a map $\tilde{f} : U_k \to V_{g(i)} \times R$. Consider the following diagram.



The two "outermost" triangles commute, while the "innermost" triangle commutes up to homotopy via the map $K : V_{g(i)} \times R \times I \to V_{g(i)} \times S^1$ defined by $K(x,r,t) = (x, \exp(rt))$ for all $(x,r,t) \in V_{g(i)} \times R \times I$. It thus follows that, as maps into U_i, $1_{U_k} \simeq \pi g_i^! f'_{g(i)}\alpha = \pi g_i^!(1 \times \exp)\tilde{f} \simeq \pi g_i^! \alpha\pi\tilde{f} = \pi g_i^! \alpha\pi(1 \times \exp)\tilde{f} = (\pi g_i^! \alpha)(\pi(f'_{g(i)}|U_k \times S^1)\alpha) = g_i f_{g(i)}|U_k$. We have thus succeeded in showing that $\underline{g}\ \underline{f} \simeq \underline{1}_{\underline{X}}$. Of course a symmetric argument shows that $\underline{f}\ \underline{g} \simeq \underline{1}_{\underline{Y}}$, and the proof is complete.

Remark. It is obvious that S^1 may be replaced in the above proof by any complex K of type $(\Pi,1)$. We need only replace $\exp: R \to S^1$ by the universal cover $P:\tilde{K} \to K$, noting that $\tilde{K}$ is contractible.

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