TOPOLOGY PROCEEDINGS

Volume 2, 1977

Pages 483–508

http://topology.auburn.edu/tp/

THE STONE-ČECH COMPACTIFICATION AND SHAPE DIMENSION

by

James Keesling

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

THE STONE-ČECH COMPACTIFICATION AND SHAPE DIMENSION

James Keesling

Introduction

This paper is a continuation of the work of Keesling [8] and Keesling and Sher [9] in studying the shape properties of the Stone-Čech compactification of a space. For a compact space X, shape theory is the study of those properties of X that can be deduced by considering the homotopy classes of mappings of X into various polyhedra P, [X,P]. Here we are concerned with $P = M^n$, a closed manifold which is a $K(\pi,1)$ and also $P = S^n \vee S^1$. By considering maps of the Stone-Čech compactification onto these polyhedra we are able to draw certain conclusions about the shape dimension of the Stone-Čech compactification. Our specific results are as follows. Let $f: X \to Y$ be a map. Then f is *homotopically onto* provided that if $g: X \to Y$ is homotopic to f, then g is onto Y. Let $n \geq 1$ be a fixed integer and let M^n be a closed manifold having the property that the universal covering space for M^n is R^n. Suppose that X is locally compact and σ-compact such that for every compact set $K \subset X$, there is a compact set $L \subset X-K$ such that $\dim L \geq n$. Then there is a map $f: \beta X \to M^n$ which is homotopically onto. Also $f|\beta X-X$ is homotopically onto M^n as well. This implies that the shape dimension of βX is at least n and that the shape dimension of $\beta X-X$ is at least n also. If X is a Lindelöf space and K is a compactum contained in $\beta X-X$, then if $\dim K \geq n$ and M^n is as above, then there is a map $f: K \to M^n$ which is homotopically onto. This implies that for compacta $K \subset \beta X - X$, dimension and shape dimension coincide. This is generally not true since the dimension of the closed unit ball B^n in R^n is n, but the shape dimension of B^n is 0.

The above results are used to show the following. For every $n \geq 2$, there is a polyhedron P_n of dimension n and a map $f: P_n \to T^n$ to the n-dimensional torus T^n such that f is homotopically onto and $f^*: H^k(T^n) \to H^k(P_n)$ is the zero homomorphism for all $k \geq 2$. Also for every $n \geq 2$, there is a metric continuum X_n such that the shape dimension of X_n is n and $\dim X_n = n$, but the shape dimension of the suspension ΣX_n is equal to 2. This last example answers a question of S. Nowak [11].

We then consider maps onto $S^n \vee S^1$ and obtain similar results. However, in this case we encounter a dimension theoretic difficulty and the results are not quite so general as are the results about mappings onto manifold $K(\pi,1)$'s.

It appears from the results in this paper and in [8] and [9] that shape theory can be a useful tool in studying the Stone-Čech compactification. The techniques and results will likely have wider application in the future.

Preliminaries

The reader is assumed to be familiar with shape theory. The paper by S. Mardešić is a good reference [12]. We will only be concerned about compact spaces in this paper, however, and thus will not need the full generality of [12] which develops shape theory for all topological spaces. The reader is assumed to be familiar with the basic properties of the Stone-Čech compactification. Either Gillman and Jerison [2] or Walker [14] would be a good reference. In dimension theory we need only the basic results which we quote from [4] and the Appendix of [10]. We now state the most important of these for easy reference. By dim X we mean the Lebesgue covering dimension.

0.1. *Theorem* ([4, 19, p. 85]). *The dimension of a normal space* X *is the supremum of the integers* n *such that there is an essential mapping onto* I^n.

Here an *essential mapping* f: $X \to I^n$ is a map such that $f|f^{-1}(\mathrm{Bd}I^n)$ cannot be extended to all of X with values in $\mathrm{Bd}I^n$. We will also need the Sum Theorem for dimension.

0.2. *Theorem* ([4, Theorem 7, p. 148]). *If* X *is a normal space with* $X = \cup_{i=1}^{\infty} X_i$ *with* X_i *closed for all* i, *then* $\dim X = \sup\{\dim X_i\}$.

We will also need the Hopf Extension Theorem.

0.3. *Theorem* ([10, 36-17, p. 207]). *Let* X *be a paracompact space of covering dimension* $\leq n+1$ *and* A *a closed subset of* X. *Let* $f: A \to S^n$. *Then* f *is extendable over* X *if and only if* $f^*(H^n(S^n)) \subset i^*(H^n(X))$ *where* $i: A \to X$ *is the inclusion map.*

If X is a compact space, then the *shape dimension* of X is the least integer n such that there is a space Y with dim Y = n such that Y shape dominates X. We will denote this by Sd X = n. If X is a compact metric space, then this is the same as K. Borsuk's fundamental dimension, Fd X. Results from [11] will be used concerning fundamental dimension. We will now state and prove a useful elementary result about shape dimension. (For a stronger result see the discussion preceding Lemma 2.3 of [15] by J. Dydak.)

0.4. *Theorem. Let* X *be a compact space with* $\mathrm{Sd}X \leq n$. *Let* $f: X \to P$ *be a map with* P *a finite polyhedron. Then there is a finite polyhedron* Q *with* $\dim Q \leq n$ *and maps* $g: X \to Q$ *and* $r: Q \to P$ *such that* $r \circ g$ *is homotopic to* f.

Proof. Let Y shape dominate X with $\dim Y = n$. Let $f: X \to P$ be a map. Let $F: X \to Y$ and $G: Y \to X$ be shape morphisms with $G \circ F = S(id_X)$. Let $f': Y \to P$ be a map such that $S(f') = S(f) \circ G$. Because Y is n-dimensional, there is an open cover $\mathcal{U}$ of Y of order at most n+1 such that the barycentric map $g_{\mathcal{U}}: Y \to N(\mathcal{U})$ to the nerve of $\mathcal{U}$ factors the map f' up to homotopy. That is, there is a map $r: N(\mathcal{U}) \to P$ such that $r \circ g_{\mathcal{U}}$ is homotopic to f'. Then $Q = N(\mathcal{U})$ will have dimension at most n. Let $g: X \to Q$ be a map such that $S(g) = S(g_{\mathcal{U}}) \circ F$. Then $S(r \circ g) = S(r) \circ S(g) = S(r) \circ S(g_{\mathcal{U}}) \circ F = S(f') \circ F = S(f) \circ G \circ F = S(f)$. Thus $S(r \circ g) = S(f)$. This implies that $r \circ g$ and f are homotopic since P is an ANR.

We let $H^n(X,A)$ denote n-dimensional Čech cohomology with integer coefficients where X is a paracompact space and A is a closed subset of X. We assume a basic knowledge of Čech cohomology. The following theorem follows from the Hopf Extension Theorem.

0.5. *Theorem. If* X *is a paracompact space of covering* *dimension* n *and* A *is a closed subset of* X, *then* $H^n(X,A)$ *is isomorphic to* $[(X,A), (S^n,p)]$.

1. Maps onto Manifold K(Π,1)'s

Let M^n be a closed n-manifold whose universal covering space is R^n. For example, M^n could be the n-dimensional torus T^n. However, many more possibilities exist, see for instance [5]. It is an unsolved problem whether the covering space of a closed n-manifold $K(\pi,1)$ must be R^n [3, §3, p. 423]. For n = 3, this is related to the Poincaré Conjecture. Our proofs seem to require that the universal covering space for M^n be R^n and so we make this assumption rather than that M^n be a closed manifold $K(\pi,1)$. In this first section of the paper we study maps from βX onto M^n. The results give us a better understanding of the shape of βX and of compacta contained in $\beta X-X$. The main results in this section are Theorems 1.2, 1.3, 1.6, and 1.8. In the next section we give some further applications of these theorems. First we prove an important lemma.

1.1. *Lemma. Suppose that* P *is a finite polyhedron and that* $H: \beta X \times I \to P$ *is a homotopy. Suppose that* $\tilde{P}$ *is the universal covering space for* P. *Let* K *be a compact set in* $\tilde{P}$. *Then there is an open set* U *in* $\tilde{P}$ *containing* K *such that the closure of* U *in* $\tilde{P}$ *is compact and such that for each* $x \in X$ *if* $h: \{x\} \times I \to \tilde{P}$ *is any lift of the path* $H|\{x\} \times I$, *then if* $h(\{x\} \times I) \cap K \neq \emptyset$, *then* $h(\{x\} \times I) \subset U$.

Proof. Suppose not. Suppose it is false for the compact set $K \subset \tilde{P}$. Let $\{U_i\}_{i=1}^{\infty}$ be a sequence of open sets in $\tilde{P}$ such that (1) $K \subset U_1$, (2) $\overline{U}_i \subset U_{i+1}$ for all i, (3) $\overline{U}_i$ compact for all i, and (4) $\tilde{P} = \cup_{i=1}^{\infty} U_i$. Since the lemma is assumed false for K, for each i there is a point $x_i \in X$ such that some lift h_i of $H|\{x_i\} \times I$ has the property that $h_i(\{x_i\} \times I) \cap K \neq \emptyset$ and $h_i(\{x_i\} \times I) \cap (\tilde{P}-U_i) \neq \emptyset$. Let $t_i(1) \in I$ be such that $h_i(x_i,t_i(1)) \in K$ and $t_i(2) \in I$ be such that $h_i(x_i, t_i(2)) \notin U_i$. Then let $x \in \beta X$ be a limit point of the set $\{x_i\}_{i=1}^{\infty}$. Let $\{i_\alpha\}$ be a subnet of the positive integers such that (1) $x_{i_\alpha} \to x$, (2) $t_{i_\alpha}(1) \to t(1) \in I$, and (3) $t_{i_\alpha}(2) \to t(2) \in I$. Such a net and points t(1) and t(2) $\in$ I can be found routinely and we omit the details. Now let O be a contractible open set in P containing H(x,t(1)) such that the covering map c: $\tilde{P} \to P$ has the property that c restricted to each component of $c^{-1}(O)$ is a homeomorphism onto O. By the continuity of H: $\beta X \times I \to P$, there must be a β such that for all $\alpha \geq \beta$, $H(\{x_{i_\alpha}\} \times I_{i_\alpha}) \subset O$ where I_{i_α} is the subinterval of I joining $t_{i_\alpha}(1)$ to t(1). So we may suppose that $H(\{x_{i_\alpha}\} \times I_{i_\alpha}) \subset O$ for all α. Now K is compact in $\tilde{P}$ and since the components of $c^{-1}(O)$ form a discrete set in $\tilde{P}$, there can be at most a finite number of these components which intersect K. Call these components $\{O_1,\cdots,O_k\}$. For one of these components O_j there must be a subnet of $\{i_\alpha\}$ such that $h_{i_\alpha}(x_{i_\alpha}, t_{i_\alpha}(1)) \in O_j$ for all i_α of the subnet. By renaming the subnet let us assume that $h_{i_\alpha}(x_{i_\alpha},t_{i_\alpha}(1)) \in O_j$ for all α. Let p: $O \to O_j$ be the inverse of $c|O_j$. Then let $B = \{x_{i_\alpha}\} \cup \{x\} \subset \beta X$ and let f: $B \to \tilde{P}$ be defined by $f \equiv p \circ H|B \times \{t(1)\}$. Then f is a continuous lift for $H|B \times \{t(1)\}$. Thus there is a unique lift of $H|B \times I$, $\tilde{H}$: $B \times I \to \tilde{P}$, such that $\tilde{H}|B \times \{t(1)\} \equiv f$. Now by the definition of $f = \tilde{H}|B \times \{t(1)\}$, $\tilde{H}(x_{i_\alpha},t(1)) \in O_j$ for all α. Also $H(\{x_{i_\alpha}\} \times I_{i_\alpha}) \subset O$ for all

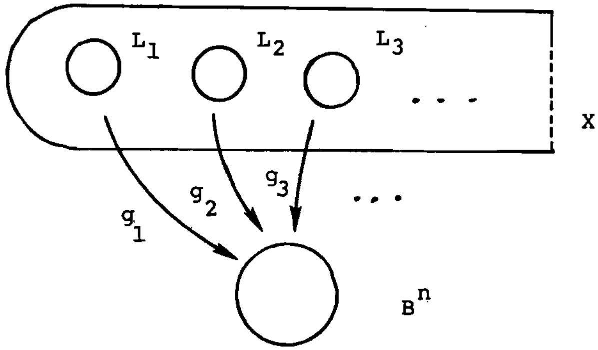
α. Thus $\tilde{H}(x_{i_\alpha}, t_{i_\alpha}(1)) \in O_j$ also. Thus $\tilde{H}|\{x_{i_\alpha}\} \times I \equiv h_{i_\alpha}$ for all α. Now $\tilde{H}(x,t(2)) \in U_i$ for some i. But since $\tilde{H}$ is continuous, there must be a β such that for all $\alpha \geq \beta$, $\tilde{H}(x_{i_\alpha}, t_{i_\alpha}(2)) \in U_i$. There is also a γ such that for all $\alpha \geq \gamma$, $i_\alpha > i$. Then let $\alpha \geq \beta$ and $\alpha \geq \gamma$. Then for this α, $\tilde{H}(x_{i_\alpha}, t_{i_\alpha}(2)) \in U_i$ and $i_\alpha > i$. This is a contradiction since $\tilde{H}(x_{i_\alpha}, t_{i_\alpha}(2)) = h_{i_\alpha}(x_{i_\alpha}, t_{i_\alpha}(2)) \notin U_{i_\alpha}$ and $\overline{U}_i \subset U_{i_\alpha}$. This contradiction proves the lemma.

We now proceed directly to the first main theorem of this section.

1.2. *Theorem. Let* M^n *be a closed* n*-manifold whose covering space is homeomorphic to* R^n. *Suppose that* X *is a locally compact* σ*-compact space such that for every compact set* $K \subset X$, *there is a compact set* $L \subset X-K$ *with dim* $L \geq n$. *Then there is a map* $f: \beta X \to M^n$ *which is homotopically onto.*

Proof. Let $X = \cup_{i=1}^{\infty} K_i$ where each K_i is compact and $K_i \subset \text{int } K_{i+1}$ for all i. Let $L_1 \subset X-K_1$ be a compact set with dim $L_1 \geq n$. Let $L_{i+1} \subset X - (\cup_{j=1}^{i} L_j \cup \cup_{j=1}^{i+1} K_j)$ with dim $L_{i+1} \geq n$. Then $\{L_i\}_{i=1}^{\infty}$ will be a sequence of disjoint compact subsets of X with dim $L_i \geq n$ for all i and with $L_i \subset X-K_i$ for all i. Now if B is any compact set in X, then there is an i with $B \subset \text{int } K_i$. Thus L_i will have the property that $L_i \subset X-B$. Now let B^n be the closed unit ball in R^n. Let $g_i: L_i \to B^n$ be an essential map (which is guaranteed to exist by Theorem 0.1). Then define $g: \cup_{i=1}^{\infty} L_i \to R^n$ by $g(x) = kg_k(x)$ for $x \in L_k$. Now $\cup_{i=1}^{\infty} L_i$ is closed in X and X is Lindelöf, hence normal. Thus there is an extension of g to all of X which we also call g. Let $c: R^n \to M^n$ be the covering map.

Then define f to be the Čech extension of $c \circ g: X \to M^n$.



Now $g: X \to R^n$ is onto and thus $f: \beta X \to M^n$ is also onto. We will now show that any map homotopic to f must also be onto.

Claim 1. *The map* $f: \beta X \to M^n$ *is homotopically onto.*

Proof of Claim 1. Suppose that $H: \beta X \times I \to M^n$ be a homotopy from f to h. Note that $g: X \to R^n$ is a lift for the map $f|X: X \to M^n$. Thus there is a unique map $\tilde{H}: X \times I \to R^n$ such that $c \circ \tilde{H} = H|X \times I$ and $\tilde{H}|X \times \{0\} \equiv g$. Let $\tilde{h} = \tilde{H}|X \times \{1\}$. Then $\tilde{h}$ is a lifting for the map $h|X$. We will show that h is onto by showing that $\tilde{h}$ is onto. Once we have shown this Claim 1 will follow and the proof of Theorem 1.2 will be complete.

Claim 2. *The map* $\tilde{h}: X \to R^n$ *is onto.*

Proof of Claim 2. Suppose that $x \in R^n$ with $x \notin \tilde{h}(X)$. Then by Lemma 1.1 there is an open set U in R^n with $x \in U$ and $\overline{U}$ compact such that if $y \in X$ and $\tilde{H}(y,t) = x$, then $\tilde{H}(\{y\} \times I) \subset U$. Let N be an integer such that $N \cdot B^n$ contains U. Then let Σ^{n-1} be the boundary of $N \cdot B^n$ and

$D_N = N \cdot g_N^{-1}(\Sigma^{n-1}) \subset L_N$. Now since $g_N: L_N \to B^n$ was an essential map, $f|D_N = N \cdot g_N|D_N$ cannot be extended to a map of L_N taking values in Σ^{n-1}. We will now get a contradiction to this. Then Claim 2 will follow.

Claim 3. *There must be an extension of* $f|D_N$ *to* L_N *taking values in* Σ^{n-1}.

Proof of Claim 3. Now x is in the interior of $N \cdot B^n$. Let $r: R^n - \{x\} \to \Sigma^{n-1}$ be the projection along the rays emanating from x. Let $s: L_N \to \Sigma^{n-1}$ be defined by $s = r \circ h$. Then s is defined and continuous since $\tilde{h}(y) \neq x$ for all $y \in X$. Consider the homotopy $\tilde{H}|D_N \times I$. Note that if $x \in \tilde{H}(D_N \times I)$, then $\tilde{H}(y,t) = x$ for some $y \in D_N$ and $t \in I$. But for that y, $\tilde{H}(y,0) \in \Sigma^{n-1}$ since $\tilde{H}(y,0) = g(y)$. This contradicts the choice of U and N, since by Lemma 1.1 $\tilde{H}(\{y\} \times I) \subset U \subset \text{int } N \cdot B^n$. Thus $x \notin \tilde{H}(D_N \times I)$. Thus $r \circ \tilde{H}|D_N \times I: D_N \times I \to \Sigma^{n-1}$ is defined and continuous and a homotopy joining the map $f|D_N$ to the map $s|D_N$. However, $s|D_N$ has a continuous extension to all of L_N having values in Σ^{n-1}, namely $r \circ \tilde{h}$. By the Borsuk Extension Theorem $f|D_N$ must also have an extension to L_N with values in Σ^{n-1}. This proves Claim 3.

Claim 3 is a contradiction of the fact that $f|L_N = N \cdot g_N$ was an essential map. This contradiction shows that $\tilde{h}$ must be onto and completes the proof of Claim 2. The fact that $\tilde{h}$ is onto shows that $h: \beta X \to M^n$ must be onto. Thus f is homotopically onto and the proof of the theorem is complete.

1.3. *Theorem. Let* M^n *be a closed* n-*manifold whose covering space is homeomorphic to* R^n. *Suppose that* X *is* *locally compact and σ-compact such that for every compact set* $K \subset X$, *there is a compact set* $L \subset X-K$ *such that dim* $L \geq n$. *Then there is a map* $f\colon \beta X-X \to M^n$ *which is homotopically onto.*

Proof. This follows from the proof of Theorem 1.2. Let $f\colon \beta X \to M^n$ be the map constructed in the proof of Theorem 1.2. We claim that $f|\beta X-X$ is homotopically onto M^n. Suppose not. Then let $h\colon \beta X-X \to M^n$ be a map homotopic to f which is not onto M^n. We may assume that $h(\beta X-X) \subset M^n-O$ where O is an open n-ball in M^n with M^n-O a manifold with boundary. Now M^n-O is an ANR. Thus there must be an extension of h to a neighborhood V of $\beta X-X$ in βX taking values in M^n-O. Call this extension h. Then $h(V) \subset M^n-O$ is also not onto M^n. We can also assume that h and $f|V$ are homotopic as maps into M^n. Since V is a neighborhood of $\beta X-X$, $X-V = K$ must be compact. Let N be such that for $i \geq N$, $L_i \cap K = \emptyset$. Then $L_i \subset V$ for all $i \geq N$. Let U be an open set containing $K \cup (\cup_{i=1}^{N-1} L_i)$ such that $\bar{U}$ is compact with $\cup_{i=N}^{\infty} L_i \subset X-U$. Then let $X' = X-U$. Then one can repeat the proof of Theorem 1.2 to show that $f|\beta X'$ must be homotopically onto M^n. However, X' is a closed subset of X which is normal and thus $cl_{\beta X}X' = \beta X'$. This gives us a contradiction since $h|cl_{\beta X}X'$ is homotopic to $f|cl_{\beta X}X'$ and $h|cl_{\beta X}X'$ is not onto. This contradiction shows that $f|\beta X-X$ must be homotopically onto. This proves Theorem 1.3.

1.4. *Corollary. Suppose that* X *is any normal space which contains a closed discrete set of compact sets* $\{L_i\}_{i=1}^{\infty}$ *such that dim* $L_i \geq n$ *for all* i. *Let* M^n *be a closed* n*-manifold* *whose universal covering space is homeomorphic to* R^n. *Then there is a map* $f: \beta X \to M^n$ *which is homotopically onto and such that* $f|\beta X - X$ *is also homotopically onto.*

1.5. *Remark.* There are other extensions of Theorems 1.2 and 1.3 along the lines of Corollary 1.4, but there is not space here to include them.

1.6. *Theorem. Let* $n \geq 1$ *be an integer. Let* X *be a locally compact and* σ*-compact space such that for every compact set* $K \subset X$ *there is a compact set* $L \subset X-K$ *such that dim* $L \geq n$. *Then* Sd $\beta X \geq n$ *and* Sd$(\beta X - X) \geq n$.

Proof. Let $T^n = M^n$ in Theorems 1.2 and 1.3. Then there is a map $f: \beta X \to T^n$ which is homotopically onto and such that $f|\beta X - X$ is also homotopically onto. Now if Sd $\beta X < n$, then by Theorem 0.4 there would be an $(n-1)$-dimensional compact polyhedron Q and maps $g: \beta X \to Q$ and $r: Q \to T^n$ such that $r \circ g$ is homotopic to f. But we can take r to be simplicial and then dim $r(Q) \leq n-1$. Thus $r \circ g(\beta X)$ would not be all of T^n. This is a contradiction. Thus Sd$(\beta X) \geq n$. Similarly Sd$(\beta X - X) \geq n$ also.

Actually we will show a much stronger result about the shape dimension of compacta in $\beta X - X$. In Corollary 1.9 we will show that for X Lindelöf and K a compactum contained in $\beta X - X$, dim K = Sd K. To prove this we will need Theorem 1.8. First we need an easy lemma.

1.7. *Lemma. Let* X *be Lindelöf and* $A \subset \beta X - X$ *be such that* A *is Lindelöf and* A *is closed in* $X \cup A$. *Then* $cl_{\beta X} A = \beta A$.

Proof. We need to show that every continuous map

$f\colon A \to [0,1]$ extends to a map $F\colon \mathrm{cl}_{\beta X}A \to [0,1]$. Let $f\colon A \to [0,1]$. Then $X \cup A$ is Lindelöf and A is closed in $X \cup A$. Thus there is an extension of f to $X \cup A$. Call this extension f again. Then f has an extension βf to βX. Then let $F = \beta f|\mathrm{cl}_{\beta X}A$. Clearly F is an extension of our original map $f\colon A \to [0,1]$.

1.8. *Theorem. Let* K *be a compactum contained in* $\beta X - X$ *where* X *is a Lindelöf space. Let* M^n *be as in Theorem* 1.2. *Then if dim* $K \geq n$, *then there is a map* $f\colon K \to M^n$ *which is homotopically onto.*

Proof. Actually we will construct a map $f\colon \beta X \to M^n$ such that $f|K$ is homotopically onto. First we will need a sequence of compact sets $L_i \subset K$ such that $L_i \cap L_j = \emptyset$ for $i \neq j$ and with $\dim L_i \geq n$ for all i.

Claim 1. *There is a sequence of compacta* $L_i \subset K$ *such that* $\{L_i\}_{i=1}^{\infty}$ *is a disjoint collection with dim* $L_i \geq n$ *for all* i.

Proof of Claim 1. Suppose that for each $x \in K$, there is a set U_x open in K containing x such that $\dim \overline{U}_x < n$. Then one could cover K by a finite number of such open sets, $\{U_1,\cdots,U_k\}$ with $\dim \overline{U}_i < n$. But then by the Sum Theorem for dimension (Theorem 0.2), $\dim K < n$, a contradiction. Thus there must be an $x_1 \in K$ such that for every open set U containing x_1, $\dim \overline{U} \geq n$. Now there must be a set U_1 open in K and containing x_1 such that $\dim K-U_1 \geq n$ (by [4, Corollary 7, p. 80]). Then let V_1 be a set containing x_1 and open in K such that $\overline{V}_1 \subset U_1$. Then $\dim \overline{V}_1 \geq n$ and $\dim(K-U_1) \geq n$ with $\overline{V}_1 \cap (K-U_1) = \emptyset$. Now let $x_2 \in K-U_1$ be such that every set U

containing x_2 and open in $K-U_1$ has the property that dim $\overline{U} \geq n$. Then let U_2 be a set open in $K-U_1$ containing x_2 such that $\dim(K-(U_1 \cup U_2)) \geq n$. Then let V_2 contain x_2 with V_2 open in $K-U_1$ such that $\overline{V}_2 \subset U_2$. Then $\overline{V}_1 \cap \overline{V}_2 = \emptyset$ and dim $\overline{V}_2 \geq n$ and $\dim(K-(U_1 \cup U_2)) \geq n$. Continuing this process one gets a sequence of sets $\{\overline{V}_i\}_{i=1}^{\infty}$ such that $\overline{V}_i \cap \overline{V}_j = \emptyset$ for $i \neq j$ and dim $\overline{V}_i \geq n$ for all i. Then $L_i = \overline{V}_i$ are the required compacta in K. This proves Claim 1.

We now resume the proof of Theorem 1.8. Let B^n be the closed unit ball in R^n and let $g_i: L_i \to B^n$ be an essential map. Let $g: \cup_{i=1}^{\infty} L_i \to R^n$ be defined by $g(x) = k \cdot g_k(x)$ for $x \in L_k$. Now $\cup_{i=1}^{\infty} L_i$ is Lindelöf. Also $K \cap (X \cup (\cup_{i=1}^{\infty} L_i)) = \cup_{i=1}^{\infty} L_i$ and thus $\cup_{i=1}^{\infty} L_i$ is closed in $X \cup (\cup_{i=1}^{\infty} L_i)$. Thus there is an extension $g: X \cup (\cup_{i=1}^{\infty} L_i) \to R^n$. Let $c: R^n \to M^n$ be the covering map and $f: \beta X \to M^n$ be the Čech extension of the map $c \circ g: X \to M^n$. Then we claim that $f|K$ is homotopically onto.

Claim 2. *The map* $f|K$ *is homotopically onto* M^n.

Proof of Claim 2. Let $D = cl_{\beta X}(\cup_{i=1}^{\infty} L_i)$. Then $D \subset K$ and actually $f|D$ is homotopically onto. The proof proceeds exactly as in the proof of Theorem 1.2. The only observation that needs to be made is that $cl_{\beta X}(\cup_{i=1}^{\infty} L_i)$ is equivalent to $\beta(\cup_{i=1}^{\infty} L_i)$ so that we may make use of Lemma 1.1. However, Lemma 1.7 shows that $cl_{\beta X}(\cup_{i=1}^{\infty} L_i) = \beta(\cup_{i=1}^{\infty} L_i)$. The proof of Claim 2 is now clear. Claim 2 completes the proof of Theorem 1.8.

1.9. *Corollary. Let* X *be a Lindelöf space and let* K *be a compactum contained in* $\beta X-X$. *Then dim* K = Sd K.

Proof. This is clear if dim K = 0 or -1. If

$\dim K \geq n \geq 1$, then by Theorem 1.8 there is a continuous map $f(K) = T^n$ which is homotopically onto. Thus $\mathrm{Sd}\, K \geq n$ also. Thus $\mathrm{Sd}\, K \geq \dim K$. However, $\mathrm{Sd}\, K \leq \dim K$ by definition. Thus $\mathrm{Sd}\, K = \dim K$.

1.10. *Example.* The above results imply that the shape dimension of βR^n and $\beta R^n - R^n$ is n and thus that the dimension of these spaces is n also. By the results of Calder and Siegel [1] we have that for $k \geq 2$, $H^k(\beta R^n) = 0 = H^k(R^n)$. Thus for each $n \geq 2$ there is a continuum $X_n = \beta R^n$ such that $\mathrm{Sd}\, X_n = n$ with $H^k(X_n) = 0$ for all $k \geq 2$. In the next section we will show that this implies the existence of a metric continuum X_n having these properties. This will provide a counterexample to a question raised by Nowak in [11].

1.11. *Example.* Let L be the long line and $X = L \times I^n$. Then $\beta(X \times I^n) = (L \cup \{\omega_1\}) \times I^n$. The space βX has trivial shape as does $\beta X - X = \{\omega_1\} \times I^n$. Thus $\mathrm{Sd}\, \beta X = 0 = \mathrm{Sd}(\beta X - X)$. However, $\dim \beta X = \dim(\beta X - X) = n$. Of course X is not σ-compact or Lindelöf. This shows that without some assumption on X one cannot expect any of the theorems in this section to hold.

2. Metric Continua and Polyhedra

In this section we apply the results of the first section to show the existence of metric continua and polyhedra which have surprising properties. One of the examples answers a question of Nowak [11]. We proceed with the examples.

2.1. *Theorem. For each* $n \geq 2$ *there is a finite* *polyhedron* P_n *of dimension* n *such that there is a map* $f: P_n \to T^n$ *with* f *homotopically onto and with* $f^*: H^k(T^n) \to H^k(P_n)$ *the zero-homomorphism for all* $k \geq 2$.

Proof. Let $X = \beta R^n$. Then there is a map $g: X \to T^n$ which is homotopically onto. Now $H^k(\beta R^n) = 0$ for all $k \geq 2$ by [1]. Thus $g^*: H^k(T^n) \to H^k(X)$ is the zero-homomorphism. Now $\dim \beta R^n = n$ and thus there is a cofinal set of finite open covers of X having order at most n+1. Let $\{\mathcal{U}_\alpha: \alpha \in A\}$ be this set. Now $H^*(X) = \varinjlim\{H^*(N(\mathcal{U}_\alpha))\}$ where $N(\mathcal{U}_\alpha)$ is the nerve of $\mathcal{U}_\alpha$ and the bonding homomorphisms are induced by the projection maps $\pi_{\alpha\beta}: N(\mathcal{U}_\beta) \to N(\mathcal{U}_\alpha)$ for $\mathcal{U}_\beta$ a refinement of $\mathcal{U}_\alpha$. Now for a cofinal collection of the α's, there are maps $g_\alpha: X \to N(\mathcal{U}_\alpha)$ and $r_\alpha: N(\mathcal{U}_\alpha) \to T^n$ such that $r_\alpha \circ g_\alpha$ is homotopic to g where $g_\alpha: X \to N(\mathcal{U}_\alpha)$ is a barycentric map for the cover $\mathcal{U}_\alpha$. Note that $r_\alpha: N(\mathcal{U}_\alpha) \to T^n$ must be homotopically onto since g is homotopically onto. Now $H^k(X) = \varinjlim\{H^k(N(\mathcal{U}_\alpha))\}$ and thus $g^* = \varinjlim\{r_\alpha: H^k(T^n) \to H^k(N(\mathcal{U}_\alpha))\}$. Since $H^k(T^n)$ is finitely generated for all k and $g^* = 0$ for all $k \geq 2$, there must be a β such that $r_\alpha^*: H^k(T^n) \to H^k(N(\mathcal{U}_\alpha))$ is the zero-homomorphism for all $\alpha \geq \beta$ and all $k \geq 2$. Then let $P_n = N(\mathcal{U}_\beta)$ and $f = r_\alpha$. This proves Theorem 2.1.

The above example has the property that for any map $g: T^n \to S^n$, $g \circ f: P_n \to S^n$ is null-homotopic. This is by Theorem 0.5 since $\dim P_n = n$. One would tend to think that the map $f: P_n \to T^n$ could not be degree zero and still be homotopically onto. However, this is in fact the case and the result is somewhat surprising.

2.2. *Theorem. Let* $n \geq 2$. *Then there is a metric* *continuum* X_n *such that* $\dim X_n = \mathrm{Sd}\, X_n = n$ *and such that* $H^k(X_n) = 0$ *for all* $k \geq 2$.

Proof. Let $Y_n = \beta R^n$. Let $g: Y_n \to T^n$ be a map which is homotopically onto. Then let $\{\mathcal{U}_\alpha : \alpha \in A\}$ be a cofinal set of open covers of Y_n such that the order of $\mathcal{U}_\alpha$ is at most n+1 for all $\alpha \in A$. There must be a $\beta \in A$ such that there are maps $r_\beta: N(\mathcal{U}_\beta) \to T^n$ and a barycentric map $g_\beta: Y_n \to N(\mathcal{U}_\beta)$ such that $r_\beta \circ g_\beta$ is homotopic to $g: Y_n \to T^n$. Then for all $\alpha \geq \beta$ the maps $g_\alpha: Y_n \to N(\mathcal{U}_\alpha)$ and $\pi_{\beta\alpha}: N(\mathcal{U}_\alpha) \to N(\mathcal{U}_\beta)$ have the property that $r_\beta \circ \pi_{\beta\alpha} \circ g_\alpha$ is homotopic to g. Thus $r_\beta \circ \pi_{\beta\alpha}: N(\mathcal{U}_\alpha) \to T^n$ must be homotopically onto since g is homotopically onto. Now $H^*(Y_n) = \varinjlim\{H^*(N(\mathcal{U}_\alpha)) : \alpha \geq \beta\}$. By an inductive process one can construct a sequence $\beta = \alpha_1 < \alpha_2 < \cdots$ in A such that $\varinjlim\{H^k(N(\mathcal{U}_{\alpha_i}))\} = 0$ for all $k \geq 2$. One uses the fact that $H^k(Y_n = 0$ for all $k \geq 2$ together with the fact that $H^k(N(\mathcal{U}_{\alpha_i}))$ is finitely generated for all i. Now let $\pi_i: N(\mathcal{U}_{\alpha_{i+1}}) \to N(\mathcal{U}_{\alpha_i})$ be a projection map for each i. Then let $X_n = \varprojlim\{N(\mathcal{U}_{\alpha_i}); \pi_i\}$. Then X_n will be a metric continuum. the reason is that each of the $N(\mathcal{U}_\alpha)$'s will be connected since βR^n is connected. Also $\dim X_n \leq n$ since $\dim N(\mathcal{U}_{\alpha_i}) \leq n$ for all i. Also $H^k(X) = \varinjlim\{H^k(\mathcal{U}_{\alpha_i}); \pi_i^*\} = 0$ for all $k \geq 2$. Now suppose that $h_i: X_n \to N(\mathcal{U}_{\alpha_i})$ are the maps making X_n the inverse limit of the inverse system $\{N(\mathcal{U}_{\alpha_i})\}$. Now consider $g_\beta \circ h_1: X_n \to T^n$. We claim that $g_\beta \circ h_1$ is homotopically onto.

Claim. The map $g_\beta \circ h_1$ *is homotopically onto.*

Proof of Claim. Suppose not. Then there must be a k such that $g_\beta \circ \pi_1 \circ \cdots \circ \pi_k: N(\mathcal{U}_{\alpha_k}) \to T^n$ is not homotopically onto. However, this implies that $g_\beta \circ \pi_{\beta\alpha_{k+1}}$ is not homotopically onto since $\pi_1 \circ \cdots \circ \pi_k$ is a projection map $\pi_{\beta\alpha_{k+1}}$ from $N(\mathcal{U}_{\alpha_{k+1}})$ to $N(\mathcal{U}_\beta)$. This is a contradiction since we have already remarked that $g_\beta \circ \pi_{\beta\alpha}$ is homotopically onto for all $\alpha \geq \beta$. Thus $g_\beta \circ h_1$ must be homotopically onto as asserted in the claim.

Now by the Claim, $g_\beta \circ h_1 \colon X_n \to T^n$ is homotopically onto. Thus $\mathrm{Sd}(X_n) = \mathrm{Fd}(X_n) \geq n$. Since $\dim X \leq n$, we must have $\dim X_n = \mathrm{Sd}\, X_n = n$. We have already shown that $H^k(X_n) = 0$ for all $k \geq 2$. Thus X_n has the desired properties and Theorem 2.2 is proved.

2.3. *Remark.* Theorem 2.2 solves Problem 6.7 in [11]. J. Hollingsworth also has an example which solves this problem.

2.4. *Corollary. Let* $n \geq 2$. *Then there exists a metric continuum* X_n *such that* $\mathrm{Sd}\, X_n = \dim X_n = n$ *with* $\mathrm{Sd}(\Sigma X_n) = 2$ *where* ΣX_n *is the suspension of* X_n.

Proof. Let X_n be the example in Theorem 2.2. Then $\mathrm{Sd}(\Sigma X_n) = 2$ by [11, Theorem 4.4].

3. Maps onto $S^n \vee S^1$

In this section we show that there are maps of βX which are homotopically onto other finite polyhedra P besides manifold $K(\pi,1)$'s. In particular we show that this is true for $P = S^n \vee S^1$. First we make the following observations about maps onto wedges of manifold $K(\pi,1)$'s. The proofs are only indicated since they are a straight forward modification of the proofs in section one.

3.1. *Theorem. Let* $n \geq 1$. *Let* $n_1, \cdots, n_k$ *be integers* *such that* $n_i \le n$ *for all* i *and suppose that* M_i *is a closed* n_i*-manifold whose covering space is* R^{n_i} *for* $i = 1,\cdots,k$. *Suppose that* X *is a locally compact* σ*-compact space such that for every compact set* $K \subset X$ *there is a compact set* $L \subset X-K$ *such that* $\dim L \ge n$. *Then there is a map* $f\colon \beta X \to \vee_{i=1}^{k} M_i$ *which is homotopically onto.*

3.2. *Theorem. Let* $n \ge 1$. *Let* $n_1,\cdots,n_k$ *be integers such that* $n_i \le n$ *for all* i *and suppose that* M_i *is a closed* n_i*-manifold whose covering space is* R^{n_i} *for* $i = 1,\cdots,k$. *Suppose that* X *is Lindelöf and that* K *is a compactum contained in* $\beta X-X$ *with* $\dim K \ge n$. *Then there is a map* $f\colon K \to \vee_{i=1}^{k} M_i$ *which is homotopically onto.*

Indication of Proof of 3.1. Let $\{L_i\}_{i=1}^{\infty}$ be disjoint compact subsets of X with $\dim L_i \ge n$ as in the proof of Theorem 1.2. Then break this up into k infinite collections: $\{L_i(1)\}_{i=1}^{\infty},\cdots,\{L_i(k)\}_{i=1}^{\infty}$. Then let $g_i(j)\colon L_i(j) \to B^{n_j}$ be an essential mapping where B^{n_j} is the closed unit ball in R^{n_j}. Then define a map $g\colon X \to \vee_{j=1}^{k} R^{n_j}$ by

$$g|L_i(j) = i\cdot g_i(j)\colon L_i(j) \to R^{n_j}$$

This defines g on $\cup_{j=1}^{k}\cup_{i=1}^{\infty} L_i(j)$ and then extend in any fashion to all of X. Then let $c_j\colon R^{n_j} \to M_j$ be the covering map for $j = 1,\cdots,k$ and let $\vee_{j=1}^{k} c_j\colon \vee_{j=1}^{k} R^{n_j} \to \vee_{j=1}^{k} M_j$ be the wedge of the c_j's. Then let $f\colon \beta X \to \vee_{j=1}^{k} M_j$ be the Čech extension of the map

$$(\vee_{j=1}^{k} c_j)\circ g\colon X \to \vee_{j=1}^{k} M_j.$$

Then one can show that f is homotopically onto in a manner similar to the proof of Theorem 1.2.

The proof of Theorem 3.2 is a similar modification of the proof of Theorem 1.8. We now proceed to maps which are homotopically onto $S^n \vee S^1$. The following proposition follows from Theorem 10.3 of [16]. We include a simple proof for completeness.

3.3. *Proposition. Let* X *be a paracompact space of dimension* n. *Then there is a closed set* $C \subset X$ *and a map* $f: (X,C) \to (B^n,S^{n-1})$ *such that* f *is not null-homotopic as a map of pairs and if* $e: (B^n,S^{n-1}) \to (S^n,p)$ *is the map which takes all of* S^{n-1} *to* p, *then* $e \circ f: (X,C) \to (S^n,p)$ *is also not null-homotopic as a map of pairs.*

Proof. Let C be a closed subset of X so chosen that there is a map $f: C \to S^{n-1}$ which cannot be extended to all of X with values in S^{n-1}. Let $i: C \to X$ be the inclusion map. Let $f: X \to B^n$ be an extension of our original map $f: C \to S^{n-1}$. Let $e: B^n \to S^n$ be the quotient map taking S^{n-1} to $p \in S^n$. Then the following diagram commutes and e^* is an isomorphism.

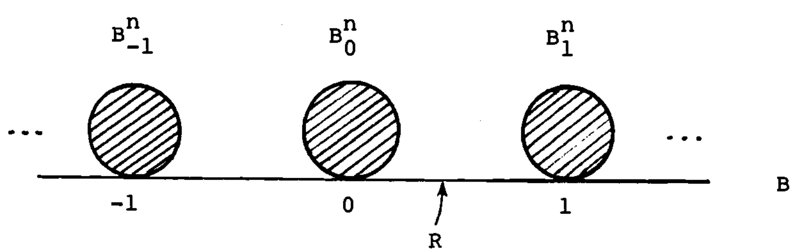
$$\begin{array}{ccccc}
H^n(S^n,p) & \xrightarrow{e^*} & H^n(B^n,S^{n-1}) & \xrightarrow{f^*} & H^n(X,C) \\
 & & \delta_1 \uparrow & & \uparrow \delta_2 \\
 & & H^{n-1}(S^{n-1}) & \xrightarrow{f^*} & H^{n-1}(C) \\
 & & & & i^* \uparrow \\
 & & & & H^{n-1}(X)
\end{array}$$

Now by the Hopf Extension Theorem (Theorem 0.3), there must be an $h \in H^{n-1}(S^{n-1})$ with $f^*(h) \notin i^*H^{n-1}(X)$ or f would have an extension to X with values in S^{n-1}. Thus $\delta_2 f^*(h) \neq 0$ in $H^n(X,C)$. Thus $f^*\delta_1(h) \neq 0$ and $f^*: H^n(B^n,S^{n-1}) \to H^n(X,C)$ is not the zero homomorphism. Thus $f: (X,C) \to (B^n,S^{n-1})$ cannot be null-homotopic. Also, since e^* is an isomorphism,

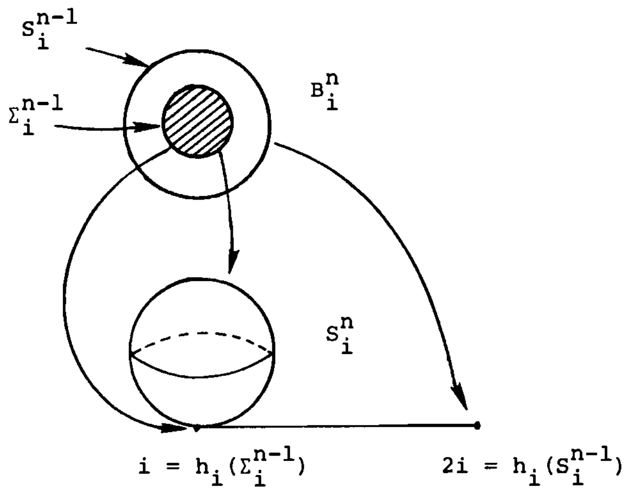
$e \circ f\colon (X,C) \to (S^n,p)$ cannot be null-homotopic either.

3.4. *Theorem. Let* $n \geq 2$. *Suppose that* X *is a locally compact* σ*-compact space such that for every compact subset* K *of* X *there is a compact set* $L \subset X$ *with* $\dim L = n$ *and with* $L \cap K = \emptyset$. *Then there is a map* f *of* βX *onto* $S^n \vee S^1$ *which is homotopically onto.*

Proof. Let $\{L_i\}_{i=1}^{\infty}$ be a sequence of compact subsets of X having the following properties: (1) $\dim L_i = n$, (2) $L_i \cap L_j = \emptyset$ for $i \neq j$, and (3) if K is a compact subset of X, then there is an i such that $L_i \cap K = \emptyset$. The construction of such a sequence of L_i's has been carried out in the proof of Theorem 1.2. Now let C_i be a closed subset of L_i and $f_i\colon (L_i,C_i) \to (B^n,S^{n-1})$ be a map of pairs which is not null-homotopic such that $e \circ f_i$ is also not null-homotopic where $e\colon (B^n,S^{n-1}) \to (S^n,p)$. Such maps f_i exist by Proposition 3.3. Now let A be the universal covering space for $S^n \vee S^1$ and $c\colon A \to S^n \vee S^1$ the covering map. We can think of A as the countable union of n-spheres attached to the real line R at the integer points. Now let B be the countable union of closed unit n-balls, B^n, attached at the integer points of R.



Now define a map $g: X \to B$ by $g|L_i = f_i$ for each i. Then let g be any extension to all of X. Such an extension exists since B is an absolute extensor. Let B_i^n be the n-ball attached at the integer point i in B. Let S_i^n be the n-sphere attached at the integer point i in A. Let $i \geq 1$ and let $h_i: B_i \to A$ be defined in the following manner. Let D_i be a closed collar for the boundary S_i^{n-1} of B_i^n and let Σ_i^{n-1} be the interior component of the boundary of D_i. Then let h_i take Σ_i^{n-1} to the point $i \in A$. Then let $h_i(S_i^{n-1})$ map to the point $2i \in A$ and let $h_i|D_i$ map onto the arc $[i,2i] \subset R \subset A$. Then let h_i map the interior of Σ_i^{n-1} homeomorphically onto $S_i^n - \{i\}$.



Then define $k_i: L_i \to A$ by $k_i = h_i \circ f_i = h_i \circ g$. Then we make the following observation.

Claim 1. *If* $m_i: L_i \to A$ *is homotopic to* k_i *by a homotopy* H *such that at each stage* t *of the homotopy* $H(C_i,t) \subset A - S_i^n$, *then* $m_i(L_i^n)$ *contains* S_i^n.

Proof of Claim 1. We can convert the homotopy H to one which maps to S_i^n by projecting all of $A - S_i^n$ to the point i. Let H' be this homotopy and let m_i' and k_i' be the corresponding ends of the homotopy. Then the homotopy is actually a homotopy of pairs $H'\colon (L_i,C_i) \times I \to (S_i^n,i)$ from k_i' to m_i'. Now k_i' is homotopic as a map of pairs to $e\circ f_i$ (or to the map $e\circ f_i$ followed by an orientation reversal of S_i^n). Thus the map m_i' is also homotopic to f_i as a map of pairs. This implies that $m_i'(L_i) = S_i^n$ and thus that $m_i(L_i)$ contains S_i^n. This proves Claim 1.

We now proceed with the proof of Theorem 3.4. Let $h\colon B \to A$ be any map such that $h|B_i = h_i$ for $i = 1,2,\cdots$. We have already defined a map $g\colon X \to B$ and the covering map $c\colon A \to S^n \vee S^1$. Then let $q\colon X \to S^n \vee S^1$ be defined by $q = c\circ h\circ g$. Then let $f = \beta q\colon \beta X \to S^n \vee S^1$. We will now show that this f is homotopically onto.

Claim 2. *If* $p\colon \beta X \to S^n \vee S^1$ *is homotopic to* f, *then* S^n *is in the image of* p.

Proof of Claim 2. Suppose that p is homotopic to f. Then $f|X$ has a lift to A, namely the map $h\circ g\colon X \to A$. Call this map $\tilde{f}$. Let H be the homotopy from f to p and let $\tilde{H}$ be the lift of this homotopy starting at $\tilde{f}$ and ending at $\tilde{p}$. Consider the point $0 \in R \subset A$. By Lemma 1.1 there is an open set U in A with $\overline{U}$ compact with $0 \in U$ such that if $x \in X$ and $h\colon \{x\} \times I \to A$ is any lift of $H|\{x\} \times I$, then if $0 \in h(\{x\} \times I)$, then $h(\{x\} \times I) \subset U$. Now let N be an integer such that if r is the projection of A onto R taking S_i^n to i for each i, then the projection of U is contained in $(-N,N)$. Then for any $x \in X$, if the homotopy $\tilde{H}$ has the property that $\tilde{H}(\{x\} \times I)$ contains $i \in R \subset A$, then $r \circ \tilde{H}(\{x\} \times I) \subset (i-N, i+N)$. Now let $M > N$. Then $h \circ g|L_M$ maps onto $S^n_M \cup [M,2M] \subset A$ with $h \circ g(C_M) = \{2M\}$. Now $\tilde{H}|L_M \times I$ must have the property that $r \circ \tilde{H}(C_M, t)$ does not contain the point M for all $t \in I$ since $2M \in \tilde{H}(C_M \times I)$ and $2M-M = M > N$. Thus $\tilde{H}(C_M \times I) \subset A - S^n_M$. Thus the map $\tilde{p}|L_M$ maps onto S^n_M by Claim 1 (since k_M in Claim 1 is just $h \circ g|L_M = \tilde{f}|L_M$). Thus $p|L_M$ maps onto S^n. Thus p contains S^n in its image. This proves Claim 2.

Claim 3. *If* $p: \beta X \to S^n \vee S^1$ *is homotopic to* f, *then* p *contains* S^1 *in its image.*

Proof of Claim 3. Let H be the homotopy joining f to p. Let $\tilde{f} = h \circ g: X \to A$ and let $\tilde{H}: X \times I \to A$ be the lift of $H|X \times I$ joining $\tilde{f}$ to $\tilde{p}$ as in Claim 2. Let N be a positive integer such that if $x \in X$ and $0 \in h(\{x\} \times I)$ for some lift of $H|\{x\} \times I$, then $r \circ h(\{x\} \times I) \subset (-N,N)$ where $r: A \to R$ is the retraction of A to R which takes each S^n_i to the point i. Let M be a positive integer with $M > 2N+1$. Then noting that $\tilde{f}(C_M) = \{2M\} \subset R$ and $\tilde{f}(y) - M$ for some $y \in L_M$ we must have that for all $x \in C_M$, $r \circ \tilde{p}(x) > 2M-N$ and for y $r \circ \tilde{p}(y) < M+N$. Thus for all $x \in C_M$, $r \circ \tilde{p}(x) - r \circ \tilde{p}(y) > 2M - N - (M+N) = M - 2N > 1$. Now by the proof of Claim 2, $\tilde{p}(L_M) \supset S^n_M$. Now we want to claim that $\tilde{p}(L_M)$ contains all of the interval $[M + N, 2M - N]$. If it does, then this interval maps onto S^1 by $c: A \to S^n \vee S^1$. Thus p will contain S^1 in its image and Claim 3 will be proved. Thus Claim 3 will follow from Claim 4.

Claim 4. *The map* $\tilde{p}|L_M$ *contains* $[M + N, 2M - N]$ *in its* *image.*

Proof of Claim 4. Suppose not and suppose that $z \in [M + N, 2M - N]$ with $z \notin \tilde{p}(L_M)$. We may assume z is not an integer point since the points in $[M + N, 2M - N]$ not in the image of $\tilde{p}|L_M$ is an open set. Then let $L_M = C \cup D$ where $C = (r\circ\tilde{p}|L_M)^{-1}(-\infty,z)$ and $D = (r\circ\tilde{p}|L_M)^{-1}(z,+\infty)$. This is a separation of L_M with $C_M \subset D$. Now since $\tilde{p}(L_M)$ contains S^n_M, it must be that $\tilde{p}(C) \supseteq S^n_M$ since $\tilde{p}(D) \cap S^n_M = \emptyset$. However, $\tilde{p}|C$ is homotopic to $\tilde{f}|C$ in A and $\tilde{f}|C\colon C \to A$ factors through B^n_M. Thus $\tilde{f}|C$ is null-homotopic and $\tilde{p}|C$ is null-homotopic. Thus $\tilde{p}|L_M$ is homotopic to a map v by a homotopy H' with the property that $v|C$ is constant and $H'|D \times I \equiv \tilde{p}|D$. But then $\tilde{f}|L_M$ is homotopic to the map v by a homotopy H" which joins the homotopies H and H'. The homotopy H" has the property that $H''(C_M \times I) \subset A-S^n_M$ and the map v does not contain S^n_M in its image. This contradicts Claim 1. This contradiction shows that $\tilde{p}(L_M)$ contains all of $[M + N, 2M - N]$ and Claim 4 is proved.

3.5. *Theorem. Let* X *be locally compact and* σ*-compact and* $n \geq 2$ *be an integer. Suppose that* X *has the property that for every compact set* $K \subset X$ *there is a compact set* $L \subset X-K$ *with dim* $L = n$. *Then there is a map* $f\colon \beta X-X \to S^n \vee S^1$ *which is homotopically onto.*

Proof. The proof is a modification of the proof of Theorem 1.3.

3.6. *Theorem. Let* K *be a compactum contained in* βX-X *where* X *is a Lindelöf space. Let* $n \geq 2$. *Then if dim* $K = n$, *then there is a map* $f\colon K \to S^n \vee S^1$ *which is homotopically onto.*

Proof. The proof is similar to the proof of Theorem 1.8.

3.7. *Question.* Is it true that if X is any paracompact space with dim $X \geq n$, then there is a closed set $C \subset X$ and a map $f: (X,C) \to (B^n, S^{n-1})$ which is not null-homotopic such that if $e: (B^n, S^{n-1}) \to (S^n, p)$ collapses S^{n-1} to p, then $e \circ f: (X,C) \to (S^n, p)$ is also not null-homotopic?

3.8. *Example.* One problem in answering Question 3.7 is that we do not have a nice relationship between maps into n-spheres and cohomology for infinite-dimensional spaces. For instance for each odd prime p Kahn [6] has given an example of an infinite-dimensional metric continuum X_p such that $H^k(X_p) = 0$ for all $k > 0$, but with X_p having essential maps onto $S^{3+i(2p-2)}$ for all $i \geq 0$.

References

[1] A. Calder and J. Siegel, *Homotopy and uniform homotopy* Trans. Amer. Math. Soc. 235 (1978), 245-269.

[2] L. Gillman and M. Jerison, *Rings of continuous functions*, Springer-Verlag, New York, 1976.

[3] A. Hattori (ed.), *Manifolds-Tokyo 1973*, University of Tokyo Press, Tokyo, 1975.

[4] J. Isbell, *Uniform spaces*, American Mathematical Society, Providence, 1964.

[5] F. E. A. Johnson, *Manifolds of homotopy type* $K(\pi,1) \cdot$ I, Proc. Camb. Phil. Soc. 70 (1971), 387-393.

[6] D. S. Kahn, *An example in Čech cohomology*, Proc. Amer. Math. Soc. 16 (1965), 584.

[7] J. Keesling, *Proper mappings and the minimum dimension of a compactification of a space*, Proc. Amer. Math. Soc. 30 (1971), 593-598.

[8] ________, *Decompositions of the Stone-Čech compactification*

which are shape equivalences, Pac. J. Math. (to appear).

[9] ________ and R. B. Sher, *Shape properties of the Stone-Čech compactification*, Gen. Top. and Its Appl. (to appear).

[10] K. Nagami, *Dimension theory*, Academic Press, New York, 1970.

[11] S. Nowak, *On the fundamental dimension of approximatively 1-connected compacta*, Fund. Math. 89 (1975), 61-79.

[12] S. Mardešić, *Shapes for topological spaces*, Gen. Top. and Its Appl. 3 (1973), 265-282.

[13] E. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

[14] R. C. Walker, *The Stone-Čech compactification*, Springer-Verlag, New York, 1974.

[15] J. Dydak, *Some remarks concerning the Whitehead theorem in shape theory*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 23 (1975), 437-445.

[16] C. H. Dowker, *Mapping theorems for noncompact spaces*, Amer. J. Math. 69 (1947), 200-242.

University of Florida

Gainesville, Florida 32601

and

University of Georgia

Athens, Georgia 30602