
TOPOLOGY PROCEEDINGS



Volume 2, 1977

Pages 509–542

<http://topology.auburn.edu/tp/>

COVERING PROPERTIES ON σ -SCATTERED SPACES

by

PETER J. NYIKOS

Topology Proceedings

Web: <http://topology.auburn.edu/tp/>

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

ISSN: 0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

COVERING PROPERTIES ON σ -SCATTERED SPACES

Peter J. Nyikos

1. Introduction

Compact scattered spaces have been studied for a comparatively long time because of their many elegant properties. It has also been known for a while that paracompact scattered spaces are nicely behaved, too. For example, the product of a paracompact scattered space and any paracompact space is paracompact [T], and every first countable paracompact scattered space is metrizable [W W_2]. These theorems are easily shown to extend to spaces which can be expressed as a countable union of closed, scattered subspaces--what I will call σ -scattered spaces in this paper.

Until now, little attention has been paid to σ -scattered spaces with covering properties which generalize these two: properties like subparacompactness, θ -refinability, screenability, and so forth. One aim of this paper is to show how many covering properties, if satisfied *hereditarily* by a σ -scattered space, are also very well behaved. The following two theorems are typical of what will be proven in the course of this paper:

Theorem 1.1. If X is a hereditarily metacompact, σ -scattered space, and Y is a [hereditarily] metacompact space, then $X \times Y$ is [hereditarily] metacompact. (And "a great many" covering properties can be put in place of metacompact here.)

Theorem 1.2. A σ -scattered space is σ -discrete if, and only if, it is θ -refinable and every point is G_δ . [Aside: although "hereditarily" does not appear in this theorem, it is clear that σ -discrete is hereditary and implies hereditarily θ -refinable.].

A number of different theories are introduced in this paper, my objective being not merely to prove theorems but to do them in the most natural setting. For example, it seems that the optimal approach to metacompact spaces is essentially the same as for screenable and meta-Lindelöf spaces, but somewhat different from the optimal theory for θ -refinable spaces and considerably different from that for weakly θ -refinable spaces. With each theory, a class of spaces is introduced which is more general than the kinds of σ -scattered spaces the theory is designed to deal with.

2. Conventions

All through this paper, "space" will mean " T_1 space." Usually this separation axiom is enough to obtain the desired results.

Let us recall two possible ways of defining a scattered space.

Definition 2.1. A space is *scattered* if it does not contain a dense-in-itself subspace.

Lemma 2.2. A space is *scattered* if, and only if, every subspace Y has a point which is isolated in Y .

However, it is a third characteristic of scattered

spaces which we will be using most often in this paper.

Notation 2.3. Let X be a topological space. Let $X^{(0)} = X$. Let $X^{(1)}$ denote the collection of non-isolated points of X (the derived set of X). With $X^{(\alpha)}$ defined for an ordinal α , let $X^{(\alpha+1)} = (X^{(\alpha)})^{(1)}$. If α is a limit ordinal, let $X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)}$.

It is easy to see that a space X is scattered if, and only if, $X^{(\alpha)}$ is empty for some α .

Definition 2.4. Let X be a scattered space. A point $x \in X$ is of level α (notation: $\ell(x) = \alpha$) if it belongs to $X^{(\alpha)} - X^{(\alpha+1)}$.

Clearly, the level of a point is a unique ordinal number, and:

Lemma 2.5. Every point p of level α has a neighborhood in which every point other than p is of a strictly lower level than α .

We will be repeatedly using the natural induction which this lemma gives us on a scattered space.

A space will be called σ -scattered if it is a countable union of closed, scattered subspaces and *weakly* σ -scattered if it is a countable union of scattered subspaces. This terminology is at variance with that of [WW₃], which uses σ -scattered to mean what we call weakly σ -scattered here. This terminology has been adopted to allow easy comparison with the concept of σ -discreteness: universal usage has it that a space is σ -discrete if it is the union of countably

many *closed* discrete subspaces. We will call a space which is a countable union of discrete subspaces *weakly σ -discrete*.

The next two sections can be read independently of each other. I have put the one on weakly σ -discrete spaces first because it seems to penetrate more quickly and deeply into the structure of the spaces involved.

3. Weakly σ -Discrete Spaces

The concept of weak σ -discreteness goes very well together with weak θ -refinability. For one thing, the proof that every weakly σ -discrete space is weakly θ -refinable entails little more than recalling the relevant definitions.

Definition 3.1. A space X is *weakly θ -refinable* if for every open cover there exists an open refinement $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ such that for each $x \in X$ there exists n such that the order of \mathcal{V}_n at x is positive but finite.

H. Bennett and D. Lutzer have given a quick proof that one gets an equivalent concept by substituting the seemingly stronger "such that $\text{ord}(x, \mathcal{V}_n) = 1$ " at the end [BL].

Lemma 3.2. Every weakly σ -discrete space is (hereditarily) weakly θ -refinable.

Proof. Let $X = \bigcup_{n=1}^{\infty} X_n$ where each X_n is discrete. If \mathcal{U} is an open cover of X , let \mathcal{V}_n be a collection of open sets such that (i) each member of \mathcal{V}_n meets X_n in exactly one point, (ii) $X_n \subset \bigcup \mathcal{V}_n$, and (iii) each member of \mathcal{V}_n is contained in some member of \mathcal{U} . Then $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ is a refinement of \mathcal{U} such that $\text{ord}(x, \mathcal{V}_n) = 1$ for each $x \in X_n$.

Obviously, weak σ -discreteness is a hereditary property,

as is σ -discreteness. The following lemma is also pretty obvious.

Lemma 3.3. The finite product of [weakly] σ -discrete spaces is [weakly] σ -discrete.

The connection with σ -scattered spaces begins with:

Theorem 3.4. A scattered space is hereditarily weakly θ -refinable if, and only if, it is weakly σ -discrete.

Proof. Sufficiency comes by Lemma 3.2. To prove necessity: the theorem is obviously true for a discrete space. Assume it has been proven for all X such that every point of X is of level $\leq \beta$ for some $\beta < \alpha$.

Case I. α is a limit ordinal, and there are no points in X of level α . Let \mathcal{U} be an open cover of X such that for each $U \in \mathcal{U}$ there exists $\beta < \alpha$ such that $U^{(\beta)} = \emptyset$. Let $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ be a refinement of \mathcal{U} such that for each $x \in X$ there exists n such that $\text{ord}(x, \mathcal{V}_n) = 1$. Let X_n be the set of all $x \in X$ such that $\text{ord}(x, \mathcal{V}_n) = 1$. Then X_n is the disjoint union of relatively open sets $X_n \cap V$ ($V \in \mathcal{V}_n$), each of which is weakly σ -discrete by the induction hypothesis. Hence X_n is weakly σ -discrete, and so is X .

Case II. There is at least one point in X of level α . Then the points of level α form a closed discrete subspace of X , and by Case I and the induction hypothesis, the rest of X is weakly σ -discrete, so that X is weakly σ -discrete.

Clearly, Theorem 3.4 remains true if "weakly σ -scattered" is substituted for "scattered," since the union of countably

many weakly σ -discrete spaces is weakly σ -discrete. In fact, we can even say:

Corollary 3.5. A space is weakly σ -discrete if, and only if, it is weakly σ -scattered and hereditarily weakly θ -refinable.

The underlined portions of the proof of Theorem 3.4 will occur several more times in this paper. Very often the most economical way of proving a theorem about scattered spaces is by this very induction method.

As one can see from $[G_1]$, it is very rare for a covering property to be inversely preserved under non-closed maps, even if the fibers (i.e. inverse images of points) are compact. But with weakly σ -discrete spaces, the fibers interfere much less with each other than in general, and:

Theorem 3.6. (1) Let $f: X \rightarrow Y$ be continuous and let Y be weakly σ -discrete. If $f^{-1}(y)$ is [hereditarily] weakly θ -refinable for all $y \in Y$, then X is [hereditarily] weakly θ -refinable.

(2) Let $f: X \rightarrow Y$ be continuous and let Y be σ -discrete. If $f^{-1}(y)$ is [hereditarily] subparacompact for all $y \in Y$, then X is [hereditarily] subparacompact.

Proof. The "hereditarily" version comes free with the other, because the restriction of f to a subspace of X satisfies the same conditions.

(1) Let $Y = \bigcup_{n=1}^{\infty} Y_n$ where Y_n is a discrete subspace of Y for all n . Let \mathcal{U} be an open cover of X and, for each $y \in Y_n$, let V_y be an open subset of X containing $f^{-1}(y)$ and missing

the rest of $f^{-1}(Y_n)$. Let $\mathcal{W}(y)$ be a relatively open cover of $f^{-1}(y)$ such that (i) each member of $\mathcal{W}(y)$ is contained in some member of \mathcal{U} and (ii) $\mathcal{W}(y) = \bigcup_{n=1}^{\infty} \mathcal{W}_n(y)$ where for each $x \in f^{-1}(y)$, there exists m such that $\text{ord}(x, \mathcal{W}_m(y)) = 1$. For each W in $\mathcal{W}(y)$, pick an open subset W' of X such that (i) $W' \cap Y = W$, (ii) $W' \subset V_y$ and (iii) W' is contained in some member of \mathcal{U} . Let $\mathcal{W}'(y)$ be the set of all such W' , and let $\mathcal{W} = \{\mathcal{W}'(y) \mid y \in Y\}$. Then \mathcal{W} is the desired refinement of \mathcal{U} .

(2) We will use the characterization of subparacompact spaces given in [B₁]: Every open cover has a σ -discrete closed refinement. Let \mathcal{U} be an open cover of X and for each fiber $f^{-1}(y)$ let $\mathcal{J}(y)$ be a σ -discrete collection of closed sets covering $f^{-1}(y)$, each member of which is contained in some member of \mathcal{U} . Then $\mathcal{U} \cup \{\mathcal{J}(y) \mid y \in Y\}$ is a σ -discrete closed refinement of \mathcal{U} .

Corollary 3.7. The product of a [weakly] σ -discrete space and a subparacompact [weakly θ -refinable] space is subparacompact [weakly θ -refinable].

Of course, one can insert "hereditarily" before "[weakly]" and "subparacompact" in both places where it occurs, and get another true result.

One might surmise from all this parallelism between "subparacompact" and "weakly θ -refinable" that the hereditarily subparacompact σ -scattered spaces are precisely the σ -discrete spaces. This is not true, however: the one-point compactification of an uncountable discrete space is scattered and hereditarily subparacompact, but not σ -discrete.

What we do have is:

Lemma 3.8. The following are equivalent on a space X .

- 1) X is σ -discrete.
- 2) X is weakly σ -discrete and every subset of X is an F_σ .
- 3) X is weakly σ -discrete and every open subset of X is an F_σ .
- 3') X is weakly σ -discrete and every closed subset of X is a G_δ .

Proof. We will show that 3) implies 1), the rest of the theorem following trivially. Let $X = \bigcup_{n=1}^{\infty} X_n$ where each X_n is a discrete subspace of X . For each $x \in X_n$, let $V(x)$ be an open subset of X containing x and missing the rest of X_n . Let $V = \bigcup \{V(x) \mid x \in X_n\}$ and let $V = \bigcup_{m=1}^{\infty} F_m$ where each F_m is closed in X . Then $X_n \cap F_m$ is a closed set for each m , since each point outside F_m has a neighborhood missing F_m and each point inside F_m is inside V and consequently has a neighborhood which meets at most one point of X_n . Thus X_n is σ -discrete, hence so is X .

There has been some interest recently in the subject of σ -discrete spaces. For example, van Douwen and Wage have shown the existence, under a set-theoretic axiom $P(c)$ which is implied by Martin's axiom, that there exists a σ -discrete, collectionwise Hausdorff, non-normal Moore space [vDW]. And P. de Caux has constructed, without any set-theoretic axioms beyond ZFC, a σ -discrete, connected, completely regular Moore space [dC]. On the other hand, it can be shown, by a standard theorem in dimension theory [N, Theorem 9-10.] that every

σ -discrete normal space is strongly zero-dimensional.

In the case of σ -scattered spaces, we can go beyond Lemma 3.8 with:

Theorem 3.9. Let X be a σ -scattered space. The following conditions are equivalent if X is regular, while 1), 2), 3) and 4) are equivalent in general.

- 1) X is weakly θ -refinable and each closed subset is a G_δ .
- 2) X is σ -discrete.
- 2') X is a σ -space.
- 2'') X is semi-stratifiable
- 3) X is θ -refinable and has a G_δ -diagonal.
- 4) X is θ -refinable and every point is a G_δ .

Proof. 1) is equivalent to 2): Every weakly θ -refinable space in which every closed set is a G_δ is hereditarily subparacompact [BL]. In particular, it is hereditarily weakly θ -refinable. If it is σ -scattered, it is weakly σ -discrete by Corollary 3.5 and σ -discrete by Lemma 3.8. The converse is trivial.

2) implies 2'): clear from the definition of σ -space, which requires the existence of a σ -locally finite network.

The sequence 2') \rightarrow 2'') \rightarrow 3) \rightarrow 4) is well-known for regular spaces, cf. [0]. Of course, 3) \rightarrow 4) is true for all spaces, while 2) \rightarrow 3) is clear from Lemma 3.3. The most complicated part of the proof is that 4) implies 2). Let $X = \bigcup_{n=1}^{\infty} X_n$ where each X_n is scattered and closed in X . We will prove that every subset of X_n is a W_δ in X and then by θ -refinability it will follow that every closed subset of

X_n is a G_δ [CCN, Theorem 2.8] in X . *A fortiori*, X_n will satisfy 1) and hence 2), and therefore X will satisfy 2).

We begin by recalling some definitions from [CCN].

Let C be a subset of X . A sequence $\mathcal{G} = \{\langle G_n, A_n, \pi_n \rangle\}_{n=1}^\infty$ is called a *sieve of C in X* if each $G_n = \{G_a \mid a \in A_n\}$ is a cover of C by sets open in X , and $\pi_n: A_{n+1} \rightarrow A_n$ is such that (i) if $a \in A_{n+1}$, then $G_a \subset G_{\pi_n(a)}$ and (ii) if $a \in A_n$, then $C \cap G_a = \bigcup \{C \cap G_{a'}, \mid \pi_n(a') = a\}$. A sequence $\{G_{a_n}\}_{n=1}^\infty$, where $a_n \in A_n$ and $\pi_n(a_{n+1}) = a_n$ is called a *thread of \mathcal{G}* . It is not required that G_a and $G_{a'}$ be distinct even if a and a' are distinct members of the same A_n . C is called a W_δ -set in X if it has a sieve \mathcal{G} in X such that the intersection of each thread of \mathcal{G} is contained in C .

Now if C is a scattered subspace of X and every point of C is a G_δ in X , then for each point p of C we can define a sequence $G_n(p)$ of open sets containing p such that (i) p is the unique point of maximal C -level in $G_n(p)$, (ii) $G_{n+1}(p) \subset G_n(p)$ for all n and (iii) $\bigcap_{n=1}^\infty G_n(p) = \{p\}$. Let $\mathcal{G}_1 = \{G_1(p) \mid p \in C\}$. In general, \mathcal{G}_n will be all possible sets of the form $G_1(p_1) \cap \dots \cap G_n(p_n) = G(p_1, \dots, p_n)$, where $p_{k+1} \in C \cap \bigcap_{i=1}^k G_i(p_i)$ for all i . [The same set may occur many times over, but that is all right.] From this it follows that $\ell(p_{k+1}) < \ell(p_k)$ whenever $p_{k+1} \neq p_k$. We let $\pi_n(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1})$.

Every thread of \mathcal{G} is indexed by initial segments of a fixed sequence of points of C , $(p_n)_{n=1}^\infty$, with the levels of the p_n forming a non-increasing sequence. By well-ordering of the levels, this sequence must become constant from some point on, which means that the intersection of each thread

is a single point of C .

Thus we have not only proven Theorem 3.9 but also

Lemma 3.10. Let X be a space and let C be a scattered subspace such that each point of C is a G_δ in X . Then each subset of C is a W_δ in X .

It is impossible to weaken the hypotheses of the theorem to read "let X be a weakly σ -scattered space" or even to "let X be a weakly σ -discrete space."

Example 3.11. The Michael line (obtained by isolating the irrational points on the real line and leaving the rationals with their usual base of neighborhoods) is an example of hereditarily *paracompact* space with a G_δ -diagonal which is weakly σ -discrete but not σ -discrete: the isolated points do not form an F_σ . If one tries to imitate for the Michael line the proof that 4) implies 2) in Theorem 3.8, say to show that \mathbb{Q} is a W_δ -set, the threads will refuse to "get stuck" upon a single point of \mathbb{Q} , and some will close down upon an irrational point.

Example 3.12. D. Burke has given [B₂] an example of a locally compact space X with a G_δ -diagonal, such that $X^{(2)} = \emptyset$ [hence X is scattered and weakly σ -discrete] which is not θ -refinable. Thus 1) of Theorem 3.9 cannot be weakened to " X is weakly θ -refinable and every point is a G_δ ." A similar example was given by J. Chaber [Ch].

Theorems 3.9 and Lemma 3.10 make a nice supplement to the following "old" results:

Theorem 3.13. (1) [WW₂] Every first countable scattered space has a λ -base.

(2) [WW₃] Every first countable σ -scattered space has a base of countable order.

From information in [WW₁] there follows:

Corollary 3.14. (1) A σ -scattered space is developable if, and only if, it is first countable and θ -refinable.

(2) A σ -scattered space is metrizable if, and only if, it is first countable and paracompact.

It is not true that every first countable scattered space is σ -discrete: the spaces of Example 3.12 give counterexamples, as does the space of countable ordinals, which is not even weakly θ -refinable.

Here is a direct consequence of Theorem 3.9 which may be of interest to normal Moore space fans:

Theorem 3.15. The existence of a scattered nonmetrizable normal Moore space is independent of the usual axioms of set theory, as is the existence of a σ -scattered nonmetrizable normal Moore space.

Proof. By Theorem 3.9, every σ -scattered Moore space is σ -discrete, and Fleissner has shown [F] that it is consistent that every normal, first countable space be collectionwise Hausdorff. Clearly, a collectionwise Hausdorff, first countable, σ -discrete space has a σ -disjoint base; and by normality, this refines to a σ -discrete base.

Conversely, the modified Pixley-Roy space given in [R, p. 21] is a metacompact, nonmetrizable, scattered normal

Moore space constructed using $MA + \neg CH$.

Another famous unsolved problem has been determined as far as σ -scattered spaces are concerned:

Theorem 3.16. $[G_2]$ Every σ -scattered stratifiable space is M_1 .

It is natural to inquire, in the wake of Theorem 3.9, whether every scattered, paracompact σ -space is stratifiable, but the answer is no: van Douwen has constructed a countable scattered regular space which is not stratifiable [vDP].

Gary Gruenhage has also provided a supplement to Theorem 3.9:

Theorem 3.17. Every σ -scattered symmetrizable space is σ -discrete.

Indeed, if we let $B(n,x) = \{y | d(x,y) < \frac{1}{n}\}$, we can define D_n to be the set of all $x \in X$ such that $B(n,x) - \{x\}$ contains only points of lower level than x . Then it is easy to prove that D_n is a closed discrete subspace of X . More generally:

Theorem 3.18. [S. W. Davis and G. Gruenhage] Every σ -scattered \mathcal{J}_r -space is σ -discrete. (A definition of \mathcal{J}_r -spaces may be found in [D] or [DGN].)

Proof. Since every closed subspace of an \mathcal{J}_r -space is \mathcal{J}_r , it is enough to let X be a scattered \mathcal{J}_r -space.

Let $x \in X$, and let $\alpha = \ell(x)$. Then $X^{(\alpha)} - \{x\}$ is closed in X , so there exists an integer $n(x)$ such that if $y \in B(n(x), x) - \{x\}$, there exists $m(y)$ with $y \notin \bigcup \{B(m(y), z) | z \in X^{(\alpha)} - \{x\}\}$. Since $\bigcap_{n=1}^{\infty} B(n, x) = \{x\}$, we may choose

$k(y) > n(x)$ such that $y \notin \bigcup \{B(k(y), z) \mid z \in X^{(\alpha)}\}$.

Let $X_1 = \{y \mid y \notin B(n(x), x) - \{x\} \text{ for any point } x\}$. Then clearly every subset of X_1 is closed, hence X_1 is closed and discrete.

For each $y \in X - X_1$, let $\alpha(y) = \min\{\alpha \mid \text{there exists } x \text{ such that } \ell(x) = \alpha \text{ and } y \in B(n(x), x) - \{x\}\}$. Let $k(y)$ be such that $y \notin \bigcup \{B(k(y), x) \mid x \in X^{(\alpha(y))}\}$, and let $D_n = \{y \mid k(y) = n\}$. Let D' be any subset of D_n ; we will show that D' is closed.

Indeed, let $z \notin D'$ and let $k > n(z)$; note that $B(k, z) \cap X^{(\ell(z))} = \{z\}$. If $y \in D' \cap B(k, z)$, then $\ell(z) \geq \alpha(y)$ which implies, by the way $k(y)$ is defined, that $k < k(y) (=n)$. Hence if $m > \max\{n(z), n\}$, then $B(m, z) \cap D' = \emptyset$.

Since every subset of D_n is closed, D_n is closed and discrete.

Here are two final tidbits of information about weakly σ -discrete spaces.

Lemma 3.19. Every weakly σ -discrete, second category space contains an isolated point.

Proof. There exists a discrete subspace D such that \bar{D} has nonempty interior U . Let $p \in D \cap U$, and let V be an open neighborhood of p missing the rest of D . Then $V \cap U$ is an open subset of U containing no point of D other than p . But $D \cap U \cap V$ is dense in $V \cap U$, hence $V \cap U = \{p\}$.

Corollary 3.20. Every weakly σ -discrete, Čech complete space is scattered.

Proof. Let X be Čech complete. If X is not scattered,

it contains a closed, dense-in-itself subspace Y , and every closed subspace of a Čech complete space is Čech complete. But by Lemma 3.19, Y cannot be weakly σ -discrete; hence neither is X .

Problem 3.21. Does there exist a nontrivial, normal, connected, weakly σ -discrete space?

As already remarked, such a space (if it exists) cannot be σ -discrete.

4. P-Canonical Covers

This section revolves around the best key I have found to understanding hereditarily metacompact scattered spaces.

The inspiration for the concept of P-canonical covers was provided by the following example. Let D_0^* denote the one-point compactification of the positive integers with the discrete topology, let D_1 denote the space of countable ordinals with the discrete topology, and let D_1^* denote either the one-point compactification or the one-point Lindelöfization of D_1 (either will work).

We will associate a certain open set with each point of $D_1^* \times D_0^*$. To an isolated point (α, n) we assign the singleton $\{(\alpha, n)\}$. To a point $(*, n)$ on the "right edge" we assign the "horizontal line" $D_1^* \times \{n\}$. To a point $(\alpha, *)$ on the "top edge" we assign the "vertical line" $\{\alpha\} \times D_0^*$. And to the corner point $(*, *)$ we assign the entire product space.

This one neighborhood assignment practically tells the whole story about the covering properties of $D_1^* \times D_0^*$ and its subspaces. It gives us an open cover which is point-finite--in fact, the union of four disjoint collections of open sets.

Any open cover \mathcal{U} of the space has a refinement whose properties are at least this "nice." We merely pick, for each point, an open neighborhood contained *both* in the neighborhood assigned to it above *and* in some member of \mathcal{U} . The fact that the neighborhood assignment was 1 - 1 plays a critical role here. Were the same open neighborhood assigned to several points, one could take an open cover, none of whose members contains all the points involved, and perhaps thereby be forced to define a refinement which is "not as nice" as the assignment we started out with.

The subspaces of $D_1^* \times D_0^*$ are also well taken care of. For a subspace Y , take for each $y \in Y$ the intersection of the neighborhood assigned above to y with Y . This new assignment does the same thing for Y that the old one did for $D_1^* \times D_0^*$. Thus $D_1^* \times D_0^*$ is hereditarily metacompact and hereditarily screenable.

As the theory unfolds, it may help to refer back to this example from time to time.

Definition 4.1. Let P be a property. A P -canonical cover of a space X is an injective neighbornet V such that $V = \{V(x) \mid x \in X\}$ satisfies P . (The notion of a neighbornet, and the notation, is due to H. Junnila.)

That is, V is a function which assigns to each point x a neighborhood $V(x)$ such that $V(x) \neq V(y)$ if $x \neq y$.

By the abuse of language, we will also refer to \hat{V} as being a P -canonical cover.

Examples we will be studying include where P is the property of being point-finite, or σ -point-finite, or

σ -disjoint, or point-countable. Here is an example which we can quickly analyze and which illustrates the general idea of "blowing up the points of a space to open sets" in a nice way, which underlies the whole idea of P-canonical covers.

Example 4.2. Let P be the property of being σ -discrete. Then a regular space has a P -canonical cover if, and only if, it is σ -discrete and paracompact. Indeed, if $\mathcal{V}_n = \{V(x) \mid x \in A_n\}$ is a discrete collection, then A_n is a closed discrete subspace of X ; so if $\dot{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ where \mathcal{V}_n is σ -discrete, then X is σ -discrete. And if \mathcal{U} is an open cover of X , we can choose for any $x \in X$ an open neighborhood $W(x) \subset V(x)$ such that $W(x)$ is contained in some member of \mathcal{U} . Then \dot{W} is a σ -discrete open refinement of \mathcal{U} . Conversely, suppose X is paracompact and $X = \bigcup_{n=1}^{\infty} X_n$ where each X_n is closed discrete in X . Since X is collectionwise normal, there exists a discrete collection $\mathcal{V}_n = \{V(x) \mid x \in X_n\}$ of open sets for each n such that $V(x)$ and $V(y)$ are distinct (hence disjoint) for distinct choices of x and y . We can also assume that $V(x) \cap (\bigcup_{i=1}^n X_i) = \emptyset$ for all $x \in X_{n+1}$. Hence $\bigcup_{n=1}^{\infty} \mathcal{V}_n$ is a P -canonical cover of X .

This example is atypical in that a scattered space can be hereditarily paracompact without having a P -canonical cover where P is " σ -discrete." [Take, for instance, the one-point compactification of an uncountable discrete space.] By way of contrast, we will show below that every hereditarily metacompact σ -scattered space has a P -canonical cover, where P is "point-finite," and in fact this characterizes

such σ -scattered spaces.

Definition 4.3. Let X be a space and let P be a property. Then X is P -refinable if every open cover of X has an open refinement satisfying P .

The terms " P -cover," " P -collection," and " P -refinement" will mean what they obviously ought to.

The proofs of the following three lemmas are routine and will be omitted.

Lemma 4.4. Suppose P is a property satisfying

(A) If \mathcal{V} is a P -canonical cover of X , and $W(x)$ is a neighborhood of x contained in $V(x)$ for all $x \in X$, then \dot{W} is a P -cover.

Then if X has a P -canonical cover, X is P -refinable, and, moreover, if $Y \subset X$, every relatively open cover of Y has a refinement which can be expanded to an open cover of X satisfying P .

Lemma 4.5. Suppose P is a property satisfying

(B) If a cover satisfies P , so does every subcollection and

(C) If a cover satisfies P , so does its trace on any subspace.

Let \mathcal{V} be an asymmetrical P -canonical cover of X and let Y be a subspace of X . If $W(y) = V(y) \cap Y$ for each $y \in Y$, then W is an asymmetrical P -canonical cover of Y .

[A neighborset V is asymmetrical if $x \in V(y)$ and $y \in V(x)$ implies $x = y$.] Every scattered space has an

asymmetrical neighbornet. More generally,

Lemma 4.6. Let X be a σ -scattered space, $X = \bigcup_{n=1}^{\infty} X_n$ where each X_n is closed in X and scattered. Then X has a neighbornet U such that

(i) If $p \in X_n$, then every point of $[X_n \cap U(p)] - \{p\}$ is of lower level in X_n than is p .

(ii) If $p \in X_n - \bigcup_{i=1}^{n-1} X_i$, then $U(p) \cap X_i = \emptyset$ for all $i < n$.

Any U satisfying (i) and (ii) is asymmetrical.

Lemma 4.7. Suppose P is a property satisfying A, B, and C above. If X has a P -canonical cover, then X is hereditarily P -refinable.

Proof. Let Y be a subspace of X and let \mathcal{U} be a relatively open cover of Y . For each $x \in X$ let $W(x)$ be an open neighborhood contained in $V(x)$, where V is P -canonical, with the condition that if $y \in Y$, $W(y) \cap Y$ will be contained in some member of \mathcal{U} . The collection $\{W(y) \cap Y \mid y \in Y\}$ is the desired cover.

The following table lists some properties P which satisfy A, B, and C.

If P is the property of being	then every space with a P -canonical cover is hereditarily
$(\sigma-)$ locally finite locally countable point-countable point-finite σ -point-finite σ -disjoint	paracompact para-Lindelöf meta-Lindelöf metacompact σ -metacompact screenable

The last four properties are different from the first two in that every σ -scattered space which is hereditarily P -refinable has a P -canonical cover. This is one of the many results catalogued below in the main theorem of this section. This theorem will be proven as stated, so that the notation is more complicated than it needs to be for, say, the case where P is "point-finite." It may be helpful to "translate" the proofs to this special case ($m = 2$, $n = \aleph_0$) on a first reading. First, we have a trivial lemma which nails down the kinds of properties we will be talking about.

Lemma 4.8. Let m and n be cardinal numbers which are either equal to 2 or infinite. Let $P(m, n)$ be the property of being the union of fewer than m collections A_α , each of which satisfies $\text{ord}(x, A_\alpha) < n$ for all x . If X is a space with an open $P(m, n)$ -cover \mathcal{U} and each member U of \mathcal{U} has an open $P(m, n)$ -cover $\mathcal{V}(U)$, then $\bigcup \{ \mathcal{V}(U) \mid U \in \mathcal{U} \}$ is an open $P(m, n)$ -cover of X .

We may assume, without loss of generality, that $m > n$ whenever m is infinite, since if $n \geq m$ in this case, $P(m, n)$ is equivalent to $P(2, n)$. Note that $P(m, n)$ always satisfies (A), (B), and (C) of Lemmas 4.4 and 4.5.

Theorem 4.9. Let P be a property of the form $P(m, n)$ as in Lemma 4.8, with at least one of m, n infinite.

(1) If X is a finite product of spaces with P -canonical covers, then X has a P -canonical cover.

(2) If X has an open P -cover \mathcal{U} , and on each member of \mathcal{U} is defined a P -canonical cover, then X has a P -canonical

cover.

(3) If X is a σ -scattered space, then X has a P -canonical cover $\leftrightarrow X$ has an asymmetrical P -canonical cover $\leftrightarrow X$ is hereditarily P -refinable.

(4) If X is a space with a P -canonical cover and Y is a [hereditarily] P -refinable space, then $X \times Y$ is [hereditarily] P -refinable.

(5) If Y has a P -canonical cover and $f: X \rightarrow Y$ is a compact map, then X is P -refinable.

Proof. (1) It is enough to show this for a twofold product. Let X_1 and X_2 have P -canonical covers V_1 and V_2 . Let $(x_1, x_2) \in X_1 \times X_2$, and let $V(x_1, x_2) = V_1(x_1) \times V_2(x_2)$.

Property P means that X_1 and X_2 can be expressed as the union of subspaces,

$$X_1 = \bigcup \{X_{1\alpha} \mid \alpha < \kappa\} \quad , \quad X_2 = \bigcup \{X_{2\alpha} \mid \alpha < \lambda\} \quad (\kappa, \lambda < m)$$

with

$$\text{ord}\{V_i(x_i) \mid x_i \in X_{i\alpha}\} < n \text{ for all } \alpha \quad (i = 1, 2).$$

Then $X = \bigcup \{X_{1\alpha} \times X_{2\beta} \mid \alpha < \kappa, \beta < \lambda\}$ and

$$\text{ord}\{V_1(x_1) \times V_2(x_2) \mid (x_1, x_2) \in X_{1\alpha} \times X_{2\beta}\} < n.$$

Also, V is injective. Hence V is P -canonical.

(2) Well-order \mathcal{U} and for each $x \in X$, let U_x be the first member of \mathcal{U} containing x . Define $V(x)$ to be the neighborhood of x associated with U_x . Since \dot{V} is a subset of the P -cover described in Lemma 4.8, it satisfies P . Moreover, V is injective, as can be seen by breaking down into the cases $U_x \neq U_y$ and $U_x = U_y$.

(3) By Lemma 4.7, every space with a P -canonical cover is hereditarily P -refinable, so it is enough to show that every hereditarily P -refinable space which is σ -scattered

has an asymmetrical P-canonical cover (equivalently, an asymmetrical neighbornet satisfying P). [Remark: the following proof is simpler if X is scattered: (2) comes more directly into play and the notation is simpler.]

Let $X = \bigcup_{n=1}^{\infty} X_n$ where each X_n is a scattered, closed subspace of X . Let U be a neighbornet on X satisfying Lemma 4.6. For each n , let $X_n^{\#} = X_n - \bigcup_{i=1}^{n-1} X_i$.

Claim. There exists on each $X_n^{\#}$ a function V_n such that (i) $V_n(x)$ is an open subset of X , (ii) $x \in V_n(x) \cap U(x)$ and (iii) $\{V_n(x) \mid x \in X_n^{\#}\}$ is a P-collection.

Once this claim is proven, we define $V(x) = V_n(x)$ for the unique n such that $x \in X_n^{\#}$. Then if $p \in X_n$, $p \notin V_m(x)$ for any $m > n$, so \dot{V} will satisfy P. By Lemma 4.6, V is an asymmetrical neighbornet, since P satisfies Lemma 4.4.

The claim will be proven by induction on the level of $X_n^{\#}$, with a format similar to that of Theorem 3.4.

It is true whenever $X_n^{\#}$ is discrete: let \mathcal{V} be an open P-cover of $X - \bigcup_{i=1}^{n-1} X_i$ refining $\{X - X_n\} \cup \{U(x) \mid x \in X_n^{\#}\}$; for $x \in X_n^{\#}$ let $V_n(x)$ be any member of \mathcal{V} containing x . Then V_n satisfies the claim.

Assume the claim has been proven for all $X_n^{\#}$ such that every point of $X_n^{\#}$ is of level $\leq \beta$ for some $\beta < \alpha$.

Case I. α is a limit ordinal, and there are no points in X of level α . Let \mathcal{W} be an open P-cover of $X - \bigcup_{i=1}^{n-1} X_i$ refining $\{X - X_n\} \cup \{U(p) \mid p \in X_n^{\#}\}$. Each member W of \mathcal{W} which meets X_n is contained in $U(p)$ for some $p \in X_n^{\#}$, which means that each point of $W \cap X_n^{\#}$ is of level $\leq \ell(p) < \alpha$ in $X_n^{\#}$. So by the induction hypothesis, we can define for each $x \in W \cap X_n^{\#}$ an open set $W(x)$ such that $x \in W(x) \subset W \cap U(x)$ and

$\{W(x) \mid x \in W \cap X_n^\# \}$ is a P-collection. Now well-order \mathcal{W} and for each $x \in X_n^\#$ let $V_n(x) = W(x)$ where W is the least member of \mathcal{W} containing x . Just as in the proof of (2), $\{V_n(x) \mid x \in X_n^\# \}$ is a P-collection satisfying the claim.

Case II. There exists at least one point in $X_n^\#$ of level α . By Case I let $V_n(x)$ be defined for all $x \in X_n^\# - X_n^{\#(\alpha)}$ to satisfy the claim (with $X_n^\# - X_n^{\#(\alpha)}$ substituted for $X_n^\#$). Let \mathcal{V} be an open P-cover of $X - \bigcup_{i=1}^{n-1} X_i$ refining $\{X - X_n\} \cup \{U(x) \mid x \in X_n^\# - X_n^{\#(\alpha)}\}$. For each $x \in X_n^{\#(\alpha)}$ let $V_n(x)$ be any member of \mathcal{V} containing x . Now V_n is defined on all of $X_n^\#$; it clearly satisfies (i) and (ii) of the claim; and it is a P-collection because it is the union of two P-collections.

(4) Let \mathcal{U} be an open cover of $X \times Y$ by sets of the form $V_1 \times V_2$, with V_1 open in X and V_2 open in Y . Let \mathcal{V} be a P-canonical cover of X . For each $x \in X$, let \mathcal{A}_x be a relatively open P-refinement on $\{x\} \times Y$ of $\mathcal{U} \cap \{x\} \times Y$. Each A in \mathcal{A}_x is a "copy" of an open subset of Y , and there exists V_A open in X so that $V_A \times A$ is contained in some member of \mathcal{U} . Let $W(A) = (V(x) \cap V_A) \times A$ and let $\mathcal{W}_x = \{W(A) \mid A \in \mathcal{A}_x\}$. Then \mathcal{W}_x is a P-collection. Since $\bigcup \mathcal{W}_x \subset V(x) \times Y$, $\bigcup \{\mathcal{W}_x \mid x \in X\}$ is a P-cover by Lemma 4.8. This shows $X \times Y$ is P-refinable. To show hereditary P-refinability if Y is hereditarily P-refinable, use the same proof, but with \mathcal{U} a relatively open cover of $Z \subset X \times Y$ by sets of the form $(V_1 \times V_2) \cap Z$, and with \mathcal{A}_x a relatively open P-refinement on $(\{x\} \times Y) \cap Z$.

(5) For each $y \in Y$ let $W[y]$ be the set $f^{-1}(V(y))$ where V is a P-canonical cover on Y . Let \mathcal{U} be an open cover of X , and for each $x \in f^{-1}(y)$ let $W(x)$ be an open neighborhood of x contained in $W[y]$ and also in some member of \mathcal{U} . Let $\mathcal{W}_1(y)$

be a finite subcollection of $\{W(x) \mid x \in f^{-1}(y)\}$ covering $f^{-1}(y)$. Then $\mathcal{W} = \cup\{\mathcal{W}_1(y) \mid y \in Y\}$ is an open P -refinement of \mathcal{U} .

It is instructive to compare the proof of (4) with the standard proof that the product of two compact spaces is compact, or that the product of a compact space and a P -refinable space is P -refinable. In every case we first take the trace of an open cover on a fiber and refine that in the manner desired. Everything works fine on the fibers, but when we "blow up" the traces into open sets, we have to be careful to do it in the right way so that the open sets coming from different fibers do not intermingle too badly.

One might try to get a common generalization of (4) and (5) along the lines of Theorem 3.6, but it cannot be done in general:

Example 4.10. Let X be the space formed by taking the Niemitzki tangent disk space [SS, Example 82] and isolating the points above the x -axis, leaving the points on the x -axis with their tangent disk neighborhoods. Let Y be any vertical line of X , and let f be the projection of X on Y . Then $f^{-1}(y)$ is a discrete subspace of X for each $y \in Y$, and f is both continuous and open. But while Y has a P -canonical cover where P is "point-finite," X is not even metacompact.

One of the more important corollaries of Theorem 4.9 is Theorem 1.1, and what we get by substituting " $P(m,n)$ -refinable" for "metacompact." However, the whole theory of P -canonical covers leaves us in the dark as to the status of:

Problem 4.11. Must the finite product of metacompact scattered spaces be metacompact?

. . . let alone the same problem with " σ -scattered" in place of "scattered," " $P(m,n)$ -refinable" in place of "metacompact," or like modifications of:

Problem 4.12. Is the product of a metacompact scattered space and a metacompact space likewise metacompact?

5. Finitary β -Spaces

It is possible to define P -canonical covers for θ -refinability, where P is the property of being a θ -cover: that is, the union of countably many covers \mathcal{U}_n such that for each point p there exists n such that $\text{ord}(p, \mathcal{U}_n)$ is finite. However, it is awkward to find a P -canonical cover for even the simplest examples. And I do not know the answer to:

Problem 5.1. Let X be a hereditarily θ -refinable scattered space. Must X have a P -canonical cover, where P is the property of being a θ -cover?

Anyway, it is more in the spirit of the concept of θ -refinability [Definition: For every open cover there is an open θ -cover refining it.] to use a sequence of neighbornets instead of a single neighbornet. This would give us a good theory, but an even simpler treatment of the subject of σ -scattered, hereditarily θ -refinable spaces can be gotten through a slight change in perspective.

Definition 5.2. A space X is a finitary β -space if for each $x \in X$ it is possible to define a sequence of open

neighborhoods $V_n(x)$ so that for each $p \in X$ there exists n such that $p \in V_n(x)$ for at most finitely many distinct $x \in X$. [Caution: This is not the same as having p be in finitely many *sets* of the form $V_n(x)$.]

We may assume without loss of generality that $V_n(x)$ contains $V_{n+1}(x)$ for all n .

A special example of a finitary β -space is a space which has a P -canonical cover V , where P is "point-finite." In this case we let $V_n(x) = V(x)$ for all n .

The usual definition of a β -space goes like this:

Definition 5.3. A space X is a β -space if for each $x \in X$ it is possible to define a sequence of open neighborhoods $V_n(x)$ so that if $\{x_n\}$ is a sequence and p is a point such that $p \in V_n(x_n)$ for all n , then $\{x_n\}$ clusters.

A finitary β -space, then, is the case where only finitely many distinct $\{x_n\}$ are involved, and clustering is immediate from this. One big difference is that the concept of a finitary β -space is clearly hereditary, whereas that of a β -space is not: every compact Hausdorff space is a β -space, but not every Tychonoff space is. For example, the Michael line is not a β -space, because the product of a metric space and a normal β -space is normal. Another "advantage" of finitary β -spaces is:

Theorem 5.4. Every finitary β -space is (hereditarily) θ -refinable.

Proof. Let \mathcal{U} be an open cover of the finitary β -space X and let $\{V_n\}_{n=1}^{\infty}$ satisfy Definition 5.2. For each $x \in X$

and each n , let $W_n(x)$ be contained in $V_n(x)$ and also in some member of \mathcal{U} . Then each family $\dot{W}_n(=\{W_n(x) \mid x \in X\})$ is a cover of X refining \mathcal{U} , and $\mathcal{W} = \bigcup_{n=1}^{\infty} \dot{W}_n$ is clearly a θ -cover.

Lemma 5.5. *Let X be a space and let $V(x)$ be an open neighborhood of x for all $x \in X$. It is possible to define $W(x)$ so that*

(i) $W(x)$ is an open neighborhood of x contained in $V(x)$ for all x

(ii) If $p \in W(x)$ and $x \in W(p)$, then both x and p are contained in $W(y)$ for infinitely many distinct y .

Notation. If $W(x)$ is a subset of X for all $x \in X$, we let $W^{-1}(x) = \{y \mid x \in W(y)\}$.

Proof of the Lemma. Let $X_0 = \{x \mid V^{-1}(x) \text{ is finite}\}$. If $x \in X_0$, define $W_0(x) = V(x) - (V^{-1}(x) - \{x\})$. Otherwise, define $W_0(x) = V(x)$. Let $X_1 = \{x \mid W_0^{-1}(x) \text{ is finite}\}$. For all $x \in X_1$, let $W_1(x) = W_0(x) - (W_0^{-1}(x) - \{x\})$. Note that $X_0 \subset X_1$ and that for all $x \in X_0$, $W_1(x) = W_0(x)$. In general, assume W_α has been defined, and let $X_{\alpha+1} = \{x \mid W_\alpha^{-1}(x) \text{ is finite}\}$. If $x \in X_{\alpha+1}$, let $W_{\alpha+1}(x) = W_\alpha(x) - (W_\alpha^{-1}(x) - \{x\})$. If $x \notin X_{\alpha+1}$, let $W_{\alpha+1}(x) = V(x)$. If α is a limit ordinal, let $W_\alpha(x) = \bigcap \{W_\beta(x) \mid \beta < \alpha\}$. Since this is actually the intersection of at most two open sets, $W_\alpha(x)$ is an open neighborhood of x . Take the smallest ordinal α so that $W_\alpha(x) = W_\beta(x)$ for all $\beta > \alpha$, and let $W(x) = W_\alpha(x)$ for all $x \in X$. Then if $W^{-1}(x)$ is finite, and $p \in W^{-1}(x)$, then $p \notin W(x) = W_{\alpha+1}(x)$, unless $p = x$, as desired.

Corollary 5.6. *If X is a finitary β -space, then it is*

possible to define $V_n(x)$ so that $\{V_n\}_{n=1}^\infty$ satisfies Definition 5.2 and, in addition, if $X_n = \{p \mid V_n^{-1}(p) \text{ is finite}\}$, then

(*) If $p \in X_n$, and if for some $x \in X$, $p \in V_n(x)$, then $x \notin V_n(p)$.

In particular, the restriction of V_n to X_n is an anti-symmetric neighbor-net of X_n .

Theorem 5.7. Every finitary β -space is weakly σ -discrete.

Proof. Let X be a finitary β -space and let $\{V_n\}_{n=1}^\infty$ and X_n be as in Corollary 5.6. For each $x \in X_n$, let $W_n(x) = \cap \{V_n(y) \mid x \in V_n(y)\}$. Then $W_n(x)$ is an open neighborhood of x such that if $p \in W_n(x)$, then $V_n^{-1}(x)$ is a proper subset of $V_n^{-1}(p)$. Indeed, $V_n^{-1}(p)$ contains p in addition to everything in $V_n^{-1}(x)$. Thus

$$D_{mn} = \{p \mid \text{card } V_n^{-1}(p) = m\}$$

is a discrete subspace of X .

As we will see below, every σ -scattered, hereditarily θ -refinable space is a finitary β -space. This does not extend to weakly σ -discrete spaces. The Michael line is weakly σ -discrete and hereditarily paracompact, but it is not a β -space, as remarked above.

A big advantage over P -canonical covers comes with:

Theorem 5.8. Let $X = \bigcup_{m=1}^\infty X_m$ where each X_m is a closed subspace of X . If X_m is a finitary β -space for all n , so is X .

Proof. Let $V_n^m(x)$ be defined with respect to X_m for $x \in X_m$, to satisfy Definition 5.2 and the condition $V_{n+1}^m(x) \subset V_n^m(x)$ for all n . For $x \in X_m$, let $W_n^m(x)$ be an open subspace

of X whose intersection with X_m is the X_m -interior of $V_n^m(x)$. For $x \notin X_m$, let $W_n^m(x) = X - X_m$. Now let $V_n(x) = \bigcap_{m=1}^n W_n^m$. To show that the conditions for a finitary β -space are satisfied, choose for each $x \in X$ an integer m such that $x \in X_m$, and then an integer $n > m$ such that $x \in V_n^m(y)$ for only finitely many $y \in X_m$.

Our next sequence of theorems is analogous to Theorem 4.9, and the proofs will only be sketched.

Theorem 5.9. A finite product of finitary β -spaces is a finitary β -space.

Proof. (outline). If X and Y are finitary β -spaces, let $V_n^m(x, y) = V_n(x) \times V_m(y)$. There is a pair (m, n) such that $V_n^{-1}(x)$ is finite and $V_m^{-1}(y)$ is finite, making $(V_n^m)^{-1}(x, y)$ finite.

Theorem 5.10. If X has an open θ -cover $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ by finitary β -spaces, then X is a finitary β -space.

Proof. Let X_{mn} be the (closed) subspace of all x in X such that $\text{ord}(x, \mathcal{U}_n) = m$. For each $x \in X_{mn}$ let $W_k(x)$ be the intersection of all $V_i(x)$ such that $i \leq k$ and $V_i(x)$ is defined with respect to those members of \mathcal{U}_n which contain x . Using the $W_k(x)$, we see that X_{mn} is a finitary β -space for all m, n . Hence X is a finitary β -space.

Theorem 5.11. Every σ -scattered, hereditarily θ -refinable space is a finitary β -space.

Proof. By Theorem 5.8, it is enough to show this for scattered spaces X .

It is obviously true for a discrete space.

Assume it has been proved for all X in which every point is of level $\leq \beta$ for some $\beta < \alpha$.

Case I. α is a limit ordinal and X has no points of level α . Let \mathcal{U} be an open cover of X such that for each $U \in \mathcal{U}$ there exists $\beta < \alpha$ such that $U^{(\beta)} = \emptyset$. Let $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ be an open θ -refinement of \mathcal{U} , and use the induction hypothesis and Theorem 5.10 to show that X is a finitary β -space.

Case II. X has at least one point of level α , and no points of higher level. On the open subspace $X - X^{(\alpha)}$, define $\mathcal{V}_n(x)$ by Case I. Let \mathcal{U} be an open cover of X such that each member contains at most one point of level α . Let $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ be an open θ -cover refining \mathcal{U} , and for each $x \in X^{(\alpha)}$ let $\mathcal{V}_n(x)$ be any member of \mathcal{V}_n containing x .

Theorem 5.12. *The product of a finitary β -space and a [hereditarily] θ -refinable space is [hereditarily] θ -refinable.*

Proof. Same as for (4) of Theorem 4.9, except that $A(x) = \bigcup_{n=1}^{\infty} A(n, x)$ is a relatively open θ -cover of $\{x\} \times Y$, and we let $\mathcal{W}_n(x) = \{W(A) \mid A \in A(n, x)\}$ and $\mathcal{W}(x) = \bigcup_{n=1}^{\infty} \mathcal{W}_n(x)$.

Theorem 5.13. *If Y is a finitary β -space and $f: X \rightarrow Y$ is a compact map, then X is θ -refinable.*

Proof. As in Theorem 4.9.(5).

Theorem 5.14. *Let X be a product of a σ -scattered, hereditarily θ -refinable space and a [hereditarily] θ -refinable space. Then X is [hereditarily] θ -refinable.*

This is a corollary of Theorems 5.11 and 5.12.

Problem 5.15. *Is the product of a θ -refinable scattered*

space and a θ -refinable space likewise θ -refinable?

6. Concluding Remarks

Scattered spaces are among the simplest of topological spaces. If there exists a scattered counterexample to a conjecture, it is often among the easiest to find. So, if nothing else, the results in this paper are useful in telling us where *not* to look for counterexamples.

I have not been successful in finding an adequate theory for σ -scattered, hereditarily $\delta\theta$ -refinable, or weakly $\delta\theta$ -refinable spaces. The biggest stumbling block is the finding of a satisfactory analogue to Theorems 4.9(2) and Theorem 5.10. Without them, one gets bogged down in Case I of the induction process showing that every σ -scattered space has the desired structure.

The problem of devising a successful theory for weakly $\bar{\theta}$ -refinable spaces and weakly $\delta\bar{\theta}$ -refinable spaces is also open. If a scattered space is hereditarily weakly $\bar{\theta}$ -refinable, for example, must its product with a weakly $\bar{\theta}$ -refinable space be likewise weakly $\bar{\theta}$ -refinable? And what can be said about σ -scattered irreducible spaces?

There are some covering properties for which no such theory as those above can be devised. For example, $D_0^* \times D_1^*$ shows us that the product of hereditarily paracompact scattered spaces need not be hereditarily normal. And $D_1^* \times D_2^*$ (where for D_2^* we add a single point to a discrete space of cardinal \aleph_2) is a product of scattered, hereditarily subparacompact spaces which is not hereditarily subparacompact: remove the corner point.

Similarly for orthocompactness: every open cover has a refinement \mathcal{V} such that for every $x \in X$ the set $\cap\{V \in \mathcal{V} \mid x \in V\}$ is open. Clearly, any space X such that $X^{(2)} = \emptyset$ is (hereditarily) orthocompact. J. Chaber has constructed such a space, called Y_4 , which is not countably metacompact [Ch]. Now, in [S] there is:

Theorem 6.1. If X is orthocompact and Y is compact, metric, and infinite, then $X \times Y$ is orthocompact iff X is countably metacompact.

Thus $Y_4 \times (\omega+1)$ is not orthocompact, even though both factors are hereditarily orthocompact and scattered.

Bibliography

- [B₁] D. K. Burke, *On subparacompact spaces*, Proc. Amer. Math. Soc. 23 (1969), 655-663.
- [B₂] _____, *A nondevelopable locally compact Hausdorff space with a G_δ -diagonal*, Gen. Top. Appl. 2 (1972), 287-291.
- [BL] H. R. Bennett and D. J. Lutzer, *A note on weak θ -refinability*, Gen. Top. Appl. 2 (1972), 49-54.
- [dC] P. de Caux, *A nondegenerate σ -discrete Moore space which is connected*, Fund. Math., to appear.
- [Ch] J. Chaber, *Metacompactness and the class MOBI*, Fund. Math. 91 (1976), 211-217.
- [CCN] _____, M. M. Ćoban, and K. Nagami, *On monotonic generalizations of Moore spaces, Čech complete spaces and p -spaces*, Fund. Math. 86 (1974), 107-119.
- [D] S. W. Davis, *On \mathcal{I}_r -spaces*, to appear.
- [DGN] _____, G. Gruenhage, and P. J. Nyikos, *G_δ -sets in symmetrizable and related spaces*, Gen. Top. Appl., to appear.
- [vDP] E. K. van Douwen and R. Pol, *Countable spaces without extension properties*, preprint.

- [vDW] E. K. van Douwen and M. Wage, *Small subsets of first countable spaces*, to appear.
- [F] W. Fleissner, *An introduction to normal Moore spaces in the constructible universe*, *Topology Proceedings* 1 (1976), 45-55.
- [G₁] R. F. Gittings, *Open mapping theory*, pp. 141-192 in: *Set-Theoretic Topology*, ed. by G. M. Reed, Academic Press, New York, San Francisco, and London, 1977.
- [G₂] G. Gruenhage, *Stratifiable σ -discrete spaces are M_1* , to appear.
- [N] K. Nagami, *Dimension Theory*, Academic Press, New York and London, 1970.
- [O] A. Okuyama, *A survey of the theory of σ -spaces*, *Gen. Top. Appl.* 1 (1971), 57-63.
- [R] M. E. Rudin, *Lectures in Set-Theoretic Topology*, AMS Regional Conference Series in Mathematics, No. 23, 1975.
- [S] B. Scott, *Toward a product theory for orthocompactness*, pp. 517-537 in: *Studies in Topology*, ed. by Stavrakis and Allen, Academic Press, New York, San Francisco, and London, 1975.
- [SS] L. A. Steen and J. A. Seebach, *Counterexamples in Topology*, Holt, Rinehart, and Winston, Inc., New York, 1970.
- [T] R. Telgarsky, *C-scattered and paracompact spaces*, *Fund. Math.* 73 (1971), 59-74.
- [WW₁] J. M. Worrell, Jr. and H. H. Wicke, *Characterizations of developable topological spaces*, *Canad. J. Math.* 17 (1965), 820-830.
- [WW₂] _____, *Topological completeness of first countable Hausdorff spaces I*, *Fund. Math.* 75 (1972), 209-222.
- [WW₃] _____, *Scattered spaces of point-countable type*, pp. 513-516 in: *General Topology and its Relations to Modern Analysis and Algebra IV, Part B*, Soc. Czech. Math. Phys., Prague, 1977.

Auburn University

Auburn, Alabama 36830

and

Ohio University Institute for Medicine and Mathematics

Athens, Ohio 45701