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1. Introduction

In many (perhaps most) instances when one is working with first countable spaces, the assumption is that at each point there is a descending sequence of neighborhoods which forms a base at the point. In this note, we discuss an obvious generalization of this, namely, we allow the character to be as large as it wishes but keep the monotonicity of the local bases.

Definition 1.1. We call a space X a lob-space provided that for each $x \in X$ there is an open neighborhood base $\mathcal{U}_x$ at x such that $\mathcal{U}_x$ is linearly ordered by reverse subset inclusion, i.e. $\mathcal{U}_x$ is a l̲inearly o̲rdered b̲ase at x. (A linear order is a reflexive, transitive, antisymmetric relation satisfying trichotomy.)

In section 2, we discuss certain properties of lob-spaces, in particular, we find that in these spaces if a point is in the closure of a union, then it is either in the closure of one of the sets or it is in the closure of the range of a choice function on the collection. This nice behavior of closures of unions leads to theorems on certain covering properties which use generalizations of Michael's cushioned refinement characterization of paracompactness. In particular, we characterize subparacompactness by seemingly weaker covering properties of this type.

In response to Smith's question of which classes of spaces other than q-spaces will support the results in [Sm], in section 3 we prove a number of theorems on preparacompact lob-spaces. Typical of these is that a regular lob-space is paracompact if and only if it is both irreducible and $\aleph$-preparacompact. Although there is much overlap and many similar theorems, we find that the class of q-spaces and the class of lob-spaces are distinct, and neither is a subclass of the other.

We repeat here some definitions which may not be familiar to all readers. The references given contain definitions but may not be the original sources.

Definition 1.2. Let X be a topological space.

a. [A_1] X is *weakly first countable* (satisfies the gf-axiom of countability) if and only if at each point $x \in X$ there is a decreasing sequence $\langle B(n,x): n \in \omega \rangle$ of (not necessarily open) subsets of X such that $U \subseteq X$ is open if and only if for each $x \in U$ there is $n_x \in \omega$ with $B(n_x,x) \subseteq U$.

b. [F] X is *sequential* if each of its sequentially closed subsets is closed.

c. [A_2] The *tightness of* X, t(X), is the smallest infinite cardinal κ such that if $A \subseteq X$ and $x \in \overline{A}$, then there is a set $C \subseteq A$ with $|C| \leq \kappa$ and $x \in \overline{C}$.

d. [A_2] The *character of* x *in* X, $\chi(x,X)$, is the smallest infinite cardinal κ such that there is a local base at x of cardinality less than or equal to κ.

e. [A_2] The *character of* X, $\chi(X)$ is $\sup\{\chi(x,X): x \in X\}$.

(We note that X is first countable if and only if $\chi(X) = \omega$.)

f. [Ju] The *pseudocharacter of* x *in* X, $\Psi(x,X)$, is the smallest infinite cardinal κ such that $\{x\}$ is the intersection of less than or equal to κ open subsets of X.

g. [G] X is a k-*space* if $A \subseteq X$ is open if and only if $A \cap K$ is open in K for each compact set $K \subseteq X$.

h. [M_2] X is a q-space if and only if at each point $x \in X$ there is a sequence $\langle N(x,n): n \in \omega \rangle$ of neighborhoods of x such that if $x_n \in N(x,n)$ for each $n \in \omega$, then $\langle x_n: n \in \omega \rangle$ has a cluster point.

We use c to denote 2^ω.

2. Properties of Lob-Spaces

The class of lob-spaces is quite large including the first countable spaces, the non-Archimedean spaces, the protometrizable spaces and the spaces having orthobases. (For definitions, see [N].)

Theorem 2.2. *If* X *is* T_1 *lob-space, then the following are equivalent:*

a) X *is first countable.*

b) X *is weakly first countable.*

c) X *is sequential.*

d) X *has countable tightness.*

e) If $x \in X$ *and* $\{x\}$ *is not open, then there is a countable set* $C \subset X - \{x\}$ *with* $x \in \overline{C}$.

f) Each point of X *is a* G_δ*-set.*

Proof. It is clear that a) ⇒ b) ⇒ c) ⇒ d) ⇒ e) and a) ⇒ f). We will prove that e) ⇒ a) and that f) ⇒ a).

e) $\Rightarrow$ a): If $x \in X$ and $\{x\}$ is open, then clearly $\chi(x,X) = \omega$. Suppose $x \in X$ and $\{x\}$ is not open. By e), there is a countable set $C = \{a_n: n \in \omega\}$ contained in $X - \{x\}$ with $x \in \overline{C}$. Let $\mathcal{U}_x$ be a linearly ordered base at x. For each $n \in \omega$, choose $G_n \in \mathcal{U}_x$ with $a_n \notin G_n$. If $\{G_n: n \in \omega\}$ is not cofinal in $\mathcal{U}_x$, then there exists $U \in \mathcal{U}_x$ with $U \subset G_n$ for every $n \in \omega$. Then $U \cap C = \emptyset$, which is impossible. Thus $\{G_n: n \in \omega\}$ is cofinal in $\mathcal{U}_x$, and therefore is a countable base at x. So $\chi(x,X) = \omega$.

f) $\Rightarrow$ a): Suppose $x \in X$. By f), we may choose a sequence $\langle G_n: n \in \omega \rangle$ of open sets with $\{x\} = \bigcap_{n\in\omega} G_n$. Choose a linearly ordered base $\mathcal{U}_x$ at x. For each $n \in \omega$, pick $B_n \in \mathcal{U}_x$ with $B_n \subset G_n$. If $\{B_n: n \in \omega\}$ is not cofinal in $\mathcal{U}_x$, then there is $B \in \mathcal{U}_x$ with $B \subset B_n \subset G_n$ for every $n \in \omega$. Then $\{x\} = B$, so $\chi(x,X) = \omega$. If $\{B_n: n \in \omega\}$ is cofinal in $\mathcal{U}_x$, then it is a countable base at x, so $\chi(x,X) = \omega$.

We state for comparison the following theorem from [N].

Theorem. [Nyikos] *Let* X *be a space with an orthobase. The following are equivalent:*

(1) X *is first countable.*

(2) Every point of X *is a* G_δ.

(3) X *is sequential.*

(4) X *is a* k-*space.*

(5) X *is a* q-*space.*

The similarity of this with 2.2 leads one to wonder if q-space or k-space can be added to 2.2. The space $\omega_1 + 1$ is a compact Hausdorff (hence k-space and q-space) lob-space which is not first countable and has no orthobase.

Upon inspection of the proof of 2.2 we see that the following corollaries have been proved.

Corollary 2.2.1. *If* X *is* T_1 *lob-space and there is a countable set* $C \subset X - \{x\}$ *with* $x \in \overline{C}$, *then* $\chi(x,X) = \omega$.

Corollary 2.2.2. *If* X *is an lob-space and* $x \in X$, *then* $\Psi(x,X) = \chi(x,X)$.

We make use of these corollaries in the next results.

Theorem 2.3. *If* X *is a* T_1 *lob-space, then* X *is sequentially compact if and only if* X *is countably compact.*

Proof. Of course, we need only show that if X is countably compact, then it is sequentially compact. Suppose $\langle x_n: n \in \omega \rangle$ is a sequence in X. If X is countably compact, then we choose a cluster point x of $\langle x_n: n \in \omega \rangle$. If there is a cofinal set $C \subset \omega$ with $x_m = x$ for each $m \in \omega$, then we may pick a constant subsequence. Otherwise, there exists a $k \in \omega$ with $x \in \overline{\{x_n: n > k\}} - \{x_n: n > k\}$. By 2.2.1, $\chi(x,X) = \omega$, and we may choose a subsequence of $\langle x_n: n \in \omega \rangle$ which converges to x.

Corollary 2.3.1. *If* X_α *is a* T_1 *countably compact lob-space for each* $\alpha \in \omega_1$, *then* $\Pi_{\alpha \in \omega_1} X_\alpha$ *is countably compact.*

Proof. This follows from the well known fact that the product of ω_1 sequentially compact spaces is countably compact.

The following is a consequence of Theorem 2 of [A_2].

Theorem. [Arhangel'skiǐ] *If* X *is a sequentially compact Hausdorff space in which each point is the intersection of* c *or fewer open sets, then* $|X| \leqq c$.

From this we have the next corollary.

Corollary 2.3.2. *If* X *is a* T_2 *countably compact lob-space and* $\Psi(X) \leq c$, *then* $|X| \leq c$.

To help study the behavior of weak covering properties in lob-spaces we now prove a theorem concerning closures of unions.

Theorem 2.4. *Suppose* $\mathcal{G}$ *is a collection of subsets of a space* $X, x \in X$ *and there is a linearly ordered base at* x. *If* $x \in \overline{\cup\mathcal{G}}$, *then either there exists* $G \in \mathcal{G}$ *with* $x \in \overline{G}$ *or there exists* $\mathcal{G}' \subset \mathcal{G}$ *and a choice function* y *on* $\mathcal{G}'$ *with* $x \in \overline{\{y(G): G \in \mathcal{G}'\}}$.

Proof. Suppose $x \notin \overline{G}$ for any $G \in \mathcal{G}$, and $\mathcal{B}$ is a linearly ordered base at x. Let < be a well ordering of $\mathcal{G}$. For each $G \in \mathcal{G}$, define $\mathcal{B}(G) = \{B \in \mathcal{B}: B \cap G \neq \emptyset$ but $B \cap G' = \emptyset$ for $G' < G\}$. Let $\mathcal{G}' = \{G \in \mathcal{G}: \mathcal{B}(G) \neq \emptyset\}$ and for each $G \in \mathcal{G}'$ choose $y(G) \in G$ so that $y(G) \in B$ for some $B \in \mathcal{B}(G)$.

We now show $x \in \overline{\{y(G): G \in \mathcal{G}'\}}$. Suppose $B \in \mathcal{B}$. Since $x \in \overline{\cup\mathcal{G}}$, we have that $\mathcal{S} = \{G \in \mathcal{G}: B \cap G \neq \emptyset\}$ is non-empty. Choose G_0 to be the first element of $\mathcal{S}$, with respect to <. Now $x \notin \overline{G}_0$ so there exists $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq X - \overline{G}_0$. Let H be the first element of $\{G \in \mathcal{G}: G \cap B_1 \neq \emptyset\}$ and note that $G_0 < H$ since $B_1 \cap G_0 = \emptyset$. Also note that $B_1 \in \mathcal{B}(H)$, so that $H \in \mathcal{G}'$. Choose $B_2 \in \mathcal{B}(H)$ so that $y(H) \in B_2$. Since $B_2 \cap G_0 = \emptyset$ and $B \cap G_0 \neq \emptyset$, then $B \nsubseteq B_2$ and hence $B_2 \subseteq B$.

Therefore y(H) $\in$ B and we are finished.

In [Ne], Nedev considered a condition on a space X where if $\mathcal{U}$ is any open cover of X, there is a refinement $\mathcal{F} = \cup_{n\in\omega}\mathcal{F}_n$ of $\mathcal{U}$ such that if $\mathcal{F}' \subset \mathcal{F}_n$ and y is a choice function on $\mathcal{F}'$ then the range of y is a closed discrete set, i.e. every open cover of X has a σ-weakly discrete refinement.

Corollary 2.4.1. *Every* T_3 *lob-space which satisfies Nedev's condition is subparacompact.*

Proof. The refinement given by Nedev's condition must be closure preserving by 2.4. Hence by 1.1 of [B] the space will be subparacompact.

In [Ba], $[D_1]$ and $[D_2]$, a generalization of the notion of a cushioned refinement has been studied. For a cardinal κ, a collection $\mathcal{V}$ of subsets of a space X is κ-weakly cushioned in a collection $\mathcal{U}$ if and only if there is a function f: $\mathcal{V} \to \mathcal{U}$ such that whenever $\mathcal{G} \subset \mathcal{V}$ with $|\mathcal{G}| \leq \kappa$ and x is choice function $\mathcal{G}$, we have $\overline{\{x(G): G \in \mathcal{G}\}} \subset \cup f(\mathcal{G})$.

Corollary 2.4.2. *If* X *is a lob-space and* $\mathcal{F}$ *is a* $|X|$-*weakly cushioned closed refinement of* $\mathcal{U}$, *then* $\mathcal{F}$ *is a cushioned refinement of* $\mathcal{U}$.

Using the fact subparacompactness is characterized by open covers having σ-cushioned closed refinements [J], we may now give the following characterization.

Corollary 2.4.3. *A regular lob-space* X *is subparacompact if and only if every open cover of* X *has a* σ-$|X|$-*weakly cushioned refinement.*

It was hoped that we could get this characterization of subparacompactness to work for property $|X|L$, which is that the σ-$|X|$-weakly cushioned collection is not a refinement of the original open cover, but rather that there is a σ-$|X|$-weakly cushioned refinement of the set of countable unions taken from the original open cover [D_1]. This will not work however, since Gary Gruenhage has constructed a space Z which has a point countable base (hence is first countable and satisfies property $|X|L$) which is not even countably θ-refinable, (Example 3.3 [DGN]).

3. Preparacompact Spaces

Definition 3.1. [Br] A T_2 space X is *preparacompact* (respectively, $\aleph$-*preparacompact*) if each open cover of X has an open refinement $\mathcal{H} = \{H_\alpha: \alpha \in A\}$ such that, if $B \subset A$ is infinite (respectively, uncountable) and if p_β and $q_\beta \in H_\beta$ for each $\beta \in B$ with $p_\alpha \neq p_\beta$ and $q_\alpha \neq q_\beta$ for $\alpha \neq \beta$, then the set $Q = \{q_\beta: \beta \in B\}$ has a limit point whenever $P = \{p_\beta: \beta \in B\}$ has a limit point. We refer to collections of these types as ppc-collections or $\aleph$-ppc-collections, respectively.

Since neither of these properties implies paracompactness, even in the presence of collectionwise normality, the special setting of q-spaces was used for their study in [Br] and [Sm].

In [Sm], it was asked in what setting other than q-spaces are the results obtained in [Sm] true. We will show that lob-spaces provide such a setting and that the class of lob-spaces and the class of q-spaces are not related by subclass inclusion.

Theorem 3.2. *Let* X *be a lob-space and let* $\mathcal{G} = \{G_\alpha: \alpha \in A\}$

be an $\aleph$*-ppc collection of open subsets of* X. *If there exists a discrete collection* $\{D_\beta: \beta \in B\}$ *of non-empty subsets of* X *such that* $D_\beta \subset G_\beta$ *for each* $\beta \in B \subset A$, *then* $\{G_\beta: \beta \in B\}$ *is either countable or closure preserving.*

Proof. Suppose B is uncountable and $\{G_\beta: \beta \in B\}$ is not closure preserving. There exists $x \in \overline{\cup\{G_\beta: \beta \in B\}} - \cup\{\overline{G}_\beta: \beta \in B\}$. By 2.4, there exists a subset $B_1 \subset B$ and a choice function y on $\{G_\beta: \beta \in B_1\}$ with $x \in \overline{\{y(G_\beta): \beta \in B_1\}}$. We select $B_2 \subset B_1$ so that $y(G_\beta) \neq y(G_\alpha)$ whenever $\alpha \neq \beta$ and $\alpha, \beta \in B_2$. Choose $q_\beta \in D_\beta$ for each $\beta \in B$. For $\beta \in B_2$, we let $p_\beta = y(G_\beta)$. For $\beta \in B - B_2$, we let $p_\beta = q_\beta$. Now $\{q_\beta: \beta \in B\}$ is a closed discrete set, but $x \in \overline{\{p_\beta: \beta \in B\}} - \{p_\beta: \beta \in B\}$. This contradicts the $\aleph$-ppc condition.

Remark. In 3.2, if $\aleph$-ppc is replaced by ppc, then the collection is closure preserving regardless of countability.

Theorem 3.3. *If* X *is a regular lob-space, then* X *is paracompact if and only if* X *is irreducible and* $\aleph$*-preparacompact.*

Proof. That paracompactness implies the other conditions is obvious. Suppose X is irreducible and $\aleph$-preparacompact and $\mathcal{U}$ is an open cover of X. Choose an open $\aleph$-ppc refinement of $\mathcal{U}$, say $\mathcal{G} = \{G_\alpha: \alpha \in A\}$. Since X is irreducible, $\mathcal{G}$ has an open refinement $\mathcal{H} = \{H_\beta: \beta \in B\}$ which covers X minimally and $H_\beta \subset G_\beta$ for $\beta \in B \subset A$. If $D_\beta = X - \cup\{H_\alpha: \alpha \in B - \{\beta\}\}$ for each $\beta \in B$, then we see that $\{D_\beta: \beta \in B\}$ is a discrete collection of closed non-empty sets with $D_\beta \subset G_\beta$ for each $\beta \in B$. By 3.2, $\{G_\beta: \beta \in B\}$ is a σ-closure preserving open refinement of $\mathcal{U}$. Hence we have proved that every open cover has a σ-closure preserving open refinement which implies paracompactness, [M_1].

Using a proof which is essentially the same as Smith's (Thm. 3.5, [Sm]) and Michael's characterization of paracompactness, we can establish the following theorem.

Theorem 3.4. *If* X *is a regular lob-space, then* X *is paracompact if and only if* X *is* $\delta\theta$*-refinable and* $\aleph$*-preparacompact.*

We would point out that this is not a corollary of 3.3 since $\delta\theta$-refinable spaces do not have to be irreducible.

We use 3.2 to get closure preserving refinements of the proper open collections to prove the following two results. We will prove the second. The proof of the first is similar although somewhat simpler.

Theorem 3.5. *If* X *is a regular preparacompact lob-space, then* X *is collectionwise Hausdorff.*

Theorem 3.6. *If* X *is a normal preparacompact lob-space, then* X *is collectionwise normal.*

Proof. Suppose X is a normal preparacompact lob-space and $\mathcal{A}$ is a discrete collection of closed subsets of X. For each $A \in \mathcal{A}$, choose U_A such that $A \subseteq U_A$ and $\overline{U}_A \cap (\cup\{A' \in \mathcal{A}: A' \neq A\}) = \emptyset$. The collection $\mathcal{U} = \{U_A: A \in \mathcal{A}\} \cup \{X - \cup\mathcal{A}\}$ is an open cover of X. We select $\mathcal{H}$ an open ppc-refinement of $\mathcal{U}$. For each $A \in \mathcal{A}$, let $\mathcal{V}_A = \{H \in \mathcal{H}: H \cap A \neq \emptyset\}$ and $V_A = \cup\mathcal{V}_A$. We will show that $\{V_A: A \in \mathcal{A}\}$ is closure preserving. Suppose not, pick $x \in \overline{\cup\{V_A: A \in \mathcal{A}'\}} - \cup\{\overline{V}_A: A \in \mathcal{A}'\}$ for some $\mathcal{A}' \subseteq \mathcal{A}$.

By 2.4, there exists $\mathcal{A}'' \subseteq \mathcal{A}'$ and a choice function y on $\{V_A: A \in \mathcal{A}''\}$ such that $x \in \overline{\{y(V_A): A \in \mathcal{A}''\}}$. For $A \in \mathcal{A}''$, pick $H_A \in \mathcal{V}_A$ such that $y(V_A) \in H_A$ and pick $z_A \in H_A \cap A$, then $\{\{z_A\}: A \in \mathcal{A}''\}$ is a discrete collection so by 3.2 $\{H_A: A \in \mathcal{A}''\}$ is closure preserving. Hence $x \in \cup\{\overline{H}_A: A \in \mathcal{A}''\} \subseteq \cup\{\overline{V}_A: A \in \mathcal{A}''\} \subseteq \cup\{\overline{V}_A: A \in \mathcal{A}'\}$, which is a contradiction. Now the collection $\{G_A: A \in \mathcal{A}\}$, where $G_A = V_A - \cup\{\overline{V}_{A'}: A' \neq A\}$, separates $\mathcal{A}$.

Using the technique presented in this section and in [Sm], we can extend these results with notions of expandability. For example if (discretely) ppc-expandable and (discretely) closure-preserving-expandable are defined in the natural way, we have that discretely ppc-expandable implies discretely closure-preserving-expandable for lob-spaces. Hence, in lob-spaces, we may factor paracompactness into a covering property and a type of expandability much as we did into a covering property and preparacompactness in this section.

Example 3.7. The space βN, the Stone-Čech compactification of the natural numbers, is a q-space which is not an lob-space. That it is a q-space is clear from compactness. That it is not an lob-space follows from 2.2 since each point of $\beta N - N$ is in the closure of N, but the space is not first countable.

Example 3.8. The space D_1^*, the one point Lindelöfization of a discrete set of cardinality ω_1, is an lob-space which is not a q-space. We may assume that the underlying set of D_1^*

is $\omega_1 + 1$ with ω_1 as the ideal point. $\{[\alpha,\omega_1]: \alpha < \omega_1\}$ is a linearly ordered base at ω_1. However, the intersection of any countable family of neighborhoods of ω_1 contains an infinite closed discrete set.

Hence there is no subclass relationship between these. However the similarity of the results in this section with those of Smith and Briggs leads one to the question of whether a theorem like 2.2 can be proved for q-spaces. There has been limited success here. Gary Gruenhage has proved the following theorem. Since this result has not appeared in print, we include a proof.

Theorem 3.9. [Gruenhage] *Every regular symmetrizable* q-*space is first countable.*

Proof. Suppose X is a regular symmetrizable q-space, and $x \in X$. Since X is regular we may choose a "q-space sequence" $\langle N(x,n): n \in \omega\rangle$ such that $\overline{N(x,n+1)} \subseteq N(x,n)$ for each $n \in \omega$. We let $N = \bigcap_{n\in\omega} N(x,n)$, and observe that N is countably compact, by the q-space condition, and closed. Hence N is a compact symmetrizable space, and thus is first countable. We now show that $\{N(x,n): n \in \omega\}$ is a base at N, and the proof will be complete. Suppose $U \subseteq X$ is open, $N \subseteq U$, and $N(x,n) - U \neq \emptyset$ for each $n \in \omega$. Choose $x_n \in N(x,n) - U$ for each $n \in \omega$. The sequence $\langle x_n: n \in \omega\rangle$ has a cluster point, say y. Since $\overline{N(x,n+1)} \subseteq N(x,n)$, we have that $y \in N(x,n)$ for every $n \in \omega$, and thus $y \in N$. But $X - U$ is closed, so $y \in X - U \subseteq X - N$, a contradiction.

The vital properties of symmetrizable spaces in the above proof are that symmetrizable is closed hereditary and that countably compact symmetrizable spaces are first countable. Hence we can weaken the hypothesis in 3.9 to $\mathcal{J}$-space or $\mathcal{J}_r$-space.

Definition 3.10. A space X is called an $\mathcal{J}$-space [HS] (respectively, $\mathcal{J}_r$-space [D_3]) if and only if there is a function B: $\omega \times X \to \mathcal{P}X$ such that the following are true:

1. For each $x \in X$, $B(n + 1,x) \subseteq B(n,x)$ for $n \in \omega$ and $\bigcap_{n\in\omega} B(n,x) = \{x\}$.
2. A set $U \subseteq X$ is open if and only if for each $x \in U$ there is $n_x \in \omega$ with $B(n_x,x) \subseteq U$.
3. If $F \subseteq X$ is closed and $x \notin F$, then there exists $n \in \omega$ such that for each $y \in B(n,x) - \{x\}$ there is $n_y \in \omega$ so that $\{x,y\} \nsubseteq \bigcup_{z\in F} B(n_y,z)$ (respectively, $y \notin \bigcup_{z\in F} B(n_y,z)$).

These spaces, and their relationship to each other, have recently been studied in [HS], [S], [D_3] and [DS]. We simply remark here that symmetrizable space implies $\mathcal{J}_r$-space implies $\mathcal{J}$-space implies weakly first countable space, and none of these is reversible. There is no chance of weakening the hypothesis of 3.9 to weakly first countable in view of Jakovlev's example [Ja] of a compact Hausdorff (hence q-space) weakly first countable space with a point which is not even a G_δ-set, using the continuum hypothesis.

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