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by

J. Dydak and J. Segal

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Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
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1. Introduction

In 1968, K. Borsuk [B_1] introduced the shape category of compact metric spaces in the pseudo-interior of the Hilbert cube Q and fundamental sequences. This version of shape theory is very close to the geometrical situation. In 1970, S. Mardešić and J. Segal [M-S_2] developed shape theory on the basis of inverse systems of ANR's. In this approach, maps between such systems are defined as well as a notion of homotopy of such maps. This homotopy relation classifies maps between ANR-systems and these classes are called shape maps. The ANR-system approach yields a continuous theory, i.e., the shape functor commutes with taking inverse limits. This is true for a single compactum or pairs of compacta. On the other hand, Borsuk's shape theory is not continuous on pairs of compacta. So while the two approaches agree on compact metric spaces, they differ on pairs of compact metric spaces. Thus the ANR-system approach is the more categorical of the two theories and permits extension to Hausdorff compacta [M-S_1] and to arbitrary topological spaces [M_1] and [Mor].

In 1972, T.A. Chapman [C_2] showed that the shape category is equivalent to the category whose objects are complements of compact Z-sets in Q and whose morphisms are weak proper homotopy classes of proper maps. Recall that a closed subset X of Q is a Z-set if for each non-empty open subset U of Q, contractible in itself, the set U-X is non-empty and contractible in itself. Moreover, all compacta in the pseudo-interior of Q are Z-sets. By a *proper* map $f: X \to Y$ is meant a map such that for every compactum $B \subset Y$ there exists a compactum $A \subset X$ such that $f(X-A) \cap B = \emptyset$. This is just a reformulation of the usual definition of proper map as a map for which $f^{-1}(B)$ is compact for any compact subset B of Y.

Chapman's approach to shape theory influenced D.A. Edwards and H.M. Hastings [E-H] to introduce the strong shape category whose objects are compact Z-sets in Q and whose morphisms are proper homotopy classes of maps of complements of compact Z-sets in Q. Their approach is quite categorical. Recently, A. Calder and H.M. Hastings [C-H] have announced that the strong shape category is precisely the quotient category obtained by inverting strong shape equivalences.

This paper is a report of our recent joint work [D-S] in which we gave a more geometrical description of the strong shape category. Our definition is equivalent to the one given in [E-H]. We emphasize geometrical conditions for a map f to induce a strong shape equivalence. By using these characterizations, we obtain results similar to those proved in [S] and announced in [E-H]. In addition to the geometrical characterizations for a map to induce a strong shape equivalence, we obtain a strong shape version of the Fox Theorem which allows us to reduce considerations to simpler cases. These results are applied to determine some important classes of maps (e.g., CE maps between compacta of finite deformation dimension or hereditary shape equivalences) which induce strong shape equivalences. Moreover, this approach allows us to answer various questions raised in $[C_1]$, $[D_2]$, and [Kod].

2. Contractible Telescopes

Throughout the paper, we assume that all (metric) spaces are locally compact and separable. Thus $X \in ANR$ means $X \in ANR(M)$ (see $[B_2]$).

If $f: X \to Y$ is a map, then by $M_n(f)$ $(n \geq 0)$ we mean the space $Y \times \{2n\} \cup X \times [2n + 1, 2n + 2]/\sim$, where $\sim$ identifies $(x, 2n + 1)$ with $(f(x), 2n)$ for each $x \in X$.

If $\underline{X} = (X_n, p_n^m)$ is an inverse sequence of compacta, then

$$\text{CTel } \underline{X} = \cup_{n=0}^{\infty} M_n(p_n^{n+1}),$$

where $p_0^1: X_1 \to X_0$ is a constant map onto a one-point space X_0. If X is a compact space, then CTel X = CTel $\underline{X}$, where $\underline{X} = (X_n, p_n^m)$, $X_n = X$ and $p_n^m = id_X$ for each $n \geq 1$ and $m \geq n$. Analogously, we can define CTel $(\underline{X,Y})$ for $(\underline{X,Y}) = ((X_n, Y_n), p_n^m)$ as the pair (CTel $\underline{X}$, CTel $\underline{Y}$).

A *net* of ANR's $\underline{U}$ is an inverse sequence (U_n, p_n^m) such that p_n^m are inclusion maps and U_n are compact ANR's.

A net of $\underline{U}$ of ANR's is said to be *coarser* than a net $\underline{V}$ of ANR's ($\underline{U} > \underline{V}$ in notation) provided $V_n \subset U_n$ and $V_n \cap \cap\underline{U} = \cap\underline{V}$ for each $n \geq 1$.

If $\underline{U} > \underline{V}$ are nets of ANR's, then

$$i(\underline{U},\underline{V}): \text{CTel } (X,Y) \to \text{CTel } (\underline{U},\underline{V})$$

denotes the inclusion map, where $X = \cap\underline{U}$ and $Y = \cap\underline{V}$.

If $f: X \to Y$ is a map of compacta, then

$$\text{CTel } f: \text{CTel } X \to \text{CTel } Y$$

is a proper map defined by

$$\mathrm{CTel}\ f(x,t) = (f(x),t) \text{ for } (x,t) \in \mathrm{CTel}\ X.$$

Recall that two maps $f, g: X \to Y$ are said to be *properly homotopic* (written $f \underset{p}{\simeq} g$) if there exists a proper map $\phi: X \times I \to Y$ such that $\phi(x,0) = f(x)$ and $\phi(x,1) = g(x)$ for each $x \in X$. If for a proper map $f: X \to Y$ there exists a proper map $g: Y \to X$ such that $g\,f \underset{p}{\simeq} id_X$ and $f\,g \underset{p}{\simeq} id_Y$ then f is said to be a *proper homotopy equivalence.*

The following can be easily derived from Lemma 3.2 of [B-S].

Theorem 2.1. *Let* A *be a closed subset of a locally compact separable space* X *and let* P *be an* ANR. *Then, for any proper map* $f: A \to P$, *there exists a proper extension* $\tilde{f}: U \to P$ *of* f *over a closed neighborhood* U *of* A *in* X.

The next theorem is a "proper" version of the Borsuk Homotopy Extension Theorem [B_2; Theorem 8.1, p. 94] whose proof is a modification of that given in [B_2] together with the use of Lemma 3.2 of [B-S].

Theorem 2.2. ([B-S; Theorem 3.1]). *Let* A *be a closed subset of a locally compact separable space* X *and let* P *be an* ANR. *Then each proper map* $H: X \times \{0\} \cup A \times I \to P$ *is properly extendible over* $X \times I$.

Theorems 2.1 and 2.2 imply that all results concerning locally compact ANR's and maps can be carried over to ANR's and proper maps. This is true since if the inclusion $i: P \to R$ of one ANR into another is a proper homotopy equivalence, then P is a strong deformation retract of R. The next two theorems are of this type.

Theorem 2.3. *Let* A *be a closed subset of an* ANR X. *If* $g: X \to X$ *is a map properly homotopic to* id_X *and* $g(a) = a$ *for* $a \in A$, *then there exists a proper map* $f: X \to X$ *such that* $f\, g \underset{p}{\simeq} id_X$ rel. A *and* $g\, f \underset{p}{\simeq} id_X$ rel. A.

In view of Theorem 2.2 the proof of Hilfssatz 2.21 of [D-K-P; p. 56] can be used to obtain a proof of Theorem 2.3. Similarly Satz 2.32 of [D-K-P; p. 64] implies the following theorem.

Theorem 2.4. *Let* A *and* B *be closed subsets of* ANR's X *and* Y *respectively. If* $g: X \to Y$ *is a proper homotopy equivalence which induces a proper homotopy equivalence* $g|A: A \to B$, *then* $g: (X,A) \to (Y,B)$ *is a proper homotopy equivalence.*

3. The Strong Shape Category

If C is a category, then $Mor_C(X,Y)$ is the set of all morphisms from X to Y in C. By $S: H_c \to Sh$ we denote the shape functor from the homotopy category to the shape category, where H_c is a full subcategory of the proper homotopy category H_p. The objects of H_c and Sh are all pairs of compact spaces. By $W_p(W_c)$, we denote a full subcategory of $H_p(H_c)$ whose objects are pairs of ANR's (pairs of compact ANR's). If (X,A) is a compact pair, then

$$\Pi_{CTel(X,A)}: W_p \to Ens$$

is the functor which assigns to each pair (P,R) of ANR's the set $Mor_{H_p}(CTel\,(X,A),\,(P,R))$.

Proposition 3.1. *Let* $\underline{U}_i > \underline{V}_i$, i = 1, 2 *be nets of* ANR's *with* $X_i = \cap\underline{U}_i$ *and* $A_i = \cap\underline{V}_i$ *for* i = 1, 2. *Then for each natural transformation*

$$\phi: \Pi_{CTel}\ (X_2,A_2) \to \Pi_{CTel}\ (X_1,A_1)$$

there exists a unique (up to proper homotopy) map

$$f: \mathrm{CTel}\ (\underline{U}_1,\underline{V}_1) \to \mathrm{CTel}\ (\underline{U}_2,\underline{V}_2)$$

such that for any proper map

$$g: \mathrm{CTel}\ (X_2,A_2) \to (P,R)$$

we have

$$\phi[g]_p = [\tilde{g}\cdot f\cdot i\ (\underline{U}_1,\underline{V}_1)]_p,$$

where $\tilde{g}$: CTel $(\underline{U}_2,\underline{V}_2) \to (P,R)$ *is any extension of* g.

If all conditions of Proposition 3.1 are satisfied, then we say that ϕ is representable by $[f]_p$. The composition of natural transformations $\phi_1\cdot\phi_2$ is representable by $[f_2f_1]_p$ where ϕ_i is representable by f_i, i = 1, 2. Now we can define the *strong shape category* s-Sh. The objects of s-Sh are pairs of compact metric spaces, a morphism from (X_1,A_1) to (X_2,A_2) is a natural transformation from $\Pi_{CTel}\ (X_2,A_2)$ to $\Pi_{CTel}\ (X_1,A_1)$. The composition of morphisms of s-Sh is the composition of the corresponding natural transformations. By Proposition 3.1 s-Sh is actually a category. Moreover, $\mathrm{Mor}_{s\text{-}Sh}((X_1,A_1),\ (X_2,A_2))$ is isomorphic to $\mathrm{Mor}_{W_p}(\mathrm{CTel}\ (\underline{U}_1,\underline{V}_1),\ \mathrm{CTel}\ (\underline{U}_2,\underline{V}_2))$.

Define the functor S^*: $H_c \to$ s-Sh as follows: $S^*(X,A) = (X,A)$ on objects and if f: $(X_1,A_1) \to (X_2,A_2)$ is a map, $S^*[f]$ is a natural transformation corresponding to $[\tilde{f}]_p$, where $\tilde{f}$: CTel $(\underline{U}_1,\underline{V}_1) \to$ CTel $(\underline{U}_2,\underline{V}_2)$ is an extension of CTel f: CTel $(X_1,A_1) \to$ CTel (X_2,A_2).

The functor Θ: s-Sh $\to$ Sh is defined as follows: $\Theta(X,A) = (X,A)$ for each compact pair (X,A). If $\phi\colon \Pi_{\mathrm{CTel}\,(X_2,A_2)} \to \Pi_{\mathrm{CTel}\,(X_1,A_1)}$ is a natural transformation, then for each pair (P,R) of compact ANR's and each map $f\colon (X_2,A_2) \to (P,R)$ there exists a unique (up to homotopy) map $f'\colon (X_1,A_1) \to (P,R)$ satisfying $[\mathrm{CTel}\ f']_p = \phi[\mathrm{CTel}\ f]_p$. Consequently, if $\Pi_{(X,A)}\colon W_c \to \mathrm{Ens}$ is the functor assigning $\mathrm{Mor}_{H_c}((X,A),\ (P,R))$ to each pair (P,R) of compact ANR's, then for each natural transformation

$$\phi\colon \Pi_{\mathrm{CTel}\,(X_2,A_2)} \to \Pi_{\mathrm{CTel}\,(X_1,A_1)}$$

we can define a natural transformation

$$\Theta(\phi)\colon \Pi_{(X_2,A_2)} \to \Pi_{(X_1,A_1)}$$

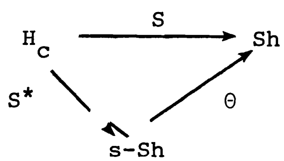
by the formula

$$\Theta(\phi)[f] = [f'], \text{ where } [\mathrm{CTel}\ f']_p = \phi[\mathrm{CTel}\ f]_p.$$

Thus we get the functor

$$\Theta\colon \text{s-Sh} \to \text{Sh}$$

satisfying $\Theta \cdot S^* = S$, i.e.,



It is shown in [E-H] that for each shape morphism $\underline{f}\colon (X,A) \to (Y,B)$ there is a strong shape morphism $\underline{\underline{g}}\colon (X,A) \to (Y,B)$ with $\Theta(\underline{\underline{g}}) = \underline{f}$. Moreover, if $\underline{f}$ is an isomorphism, then $\underline{\underline{g}}$ can be chosen to be an isomorphism of s-Sh. However, Θ is not an isomorphism of categories. Take two points x and y belonging to different composants of the dyadic solenoid D. Let $A = \{x,y\}$ and let $i\colon A \to D$ and $f\colon A \to D$ be an inclusion and constant map respectively. Then

$$\Theta S^*[i] = S[i] = S[f] = \Theta S^*[f].$$

However, $S^*[i] \neq S^*[f]$. Indeed, $S^*[i] = S^*[f]$ would imply the existence of a proper map

$$H: \mathrm{CTel}\ I \to \mathrm{CTel}\ \underline{U}$$

where $I = [0,1]$ and $D = \cap\underline{U} = \cap_{n=1}^{\infty} U_n$, such that $H(0,t) = (x,t)$ and $H(1,t) = (y,t)$ for each $t \geq 0$. Hence there would exist paths $\omega_n: I \to U_n$ joining x and y with $\omega_{n+1} = \omega_n$ in U_n rel $\{0,1\}$. This would mean that x and y are joinable in D in the sense of Krasinkiewicz-Minc (see [K-M]). But this would contradict the result of [K-M] that no two points from different composants are joinable in D.

4. Geometric Characterization of Maps Inducing Strong Shape Equivalences

Theorem 4.1. *Let* $f: X \to Y$ *be a map of compacta. The following conditions are equivalent:*

1) f *induces a strong shape equivalence,*

2) for some net $\underline{U}$ *of* ANR's *with* $\cap\underline{U} = X$ *the natural projections* $p_n: U_n \to U_n \cup_f Y$ *induce shape equivalences,*

3) for each compactum Z *containing* X *the natural projection* $p: Z \to Z\cup_f Y$ *induces a shape equivalence.*

The next theorem answers problem (SC6) in $[C_1]$.

Theorem 4.2. *Let* $X \subset Y$ *be compacta. Then the following conditions are equivalent:*

1) The inclusion map $i: X \to Y$ *induces a strong shape equivalence,*

2) for each neighborhood V *of* Y *in* Q *and for every* *neighborhood* U *of* X *in* Q *there is a homotopy* f_t: Y → V, $0 \le t \le 1$, *such that* $f_0(y) = y$ *for all* $y \in Y$, $f_1(Y) \subset U$ *and* $f_t(x) = x$ *for all* $x \in X$ *and* $0 \le t \le 1$.

3) *the inclusion* i: (X,X) → (Y,X) *is a shape equivalence.*

The Fox Theorem in Shape Theory:

Let $\underline{\underline{f}}$: $(X_1,X_0) \to (Y_1,Y_0)$ *be a strong shape morphism. Then there exist a pair* (Z_1,Z_0) *and homeomorphisms* i: $(X_1,X_0) \to (Z_1,Z_0)$ *and* j: $(Y_1,Y_0) \to (Z_1,Z_0)$ *into* (Z_1,Z_0) *such that*

1) $S^*[j]\underline{\underline{f}} = S^*[i]$,

2) $S^*[j]$ *is an isomorphism,*

3) $\dim Z_k = max(\dim Y_k, 1 + \dim X_k)$ *for* $k = 0, 1$,

4) $Z_0 \cap i(X_1) = i(X_0)$, $Z_0 \cap j(Y_1) = j(Y_0)$,

5) *if* X_0 *and* Y_0 *are one-point spaces, then* Z_0 *is an one-point space, too.*

Theorem 4.4. *Let* f: X → Y *be a map of compacta. Then the following conditions are equivalent:*

1) f *induces the strong shape equivalence* $S^*[f]$,

2) *the inclusion* i: (X,X) → (M(f),X) *induces the shape equivalence* S[i].

3) f: X → Y *and* $\hat{f}$: $\hat{M}(f) \to Y$ *induce shape equivalences.*

In the above theorem $\hat{M}(f)$ denotes the double mapping cylinder of f: X → Y which is defined to be the adjunction space

$$\hat{M}(f) = (X \times [-1,1]) \cup_\phi (Y \times \{-1,1\})$$

where

$$\phi = f \times 1: X \times \{-1,1\} \to Y \times \{-1,1\}.$$

The image of (u,t) under the quotient map $q: (X \times [-1,1]) + (Y \times \{-1,1\}) \to \hat{M}(f)$ is denoted by $[u,t]$. The map $\hat{f}: \hat{M}(f) \to Y$ is defined by $\hat{f}[u,t] = f(u)$, if $(u,t) \in X \times [-1,1]$ and by $\hat{f}[u,t] = u$, if $(u,t) \in Y \times \{-1,1\}$. This result is obtained from Theorem 4.1 by using a technique due to Kozlowski [K].

5. Some Classes of Maps Which Induce Strong Shape Equivalences

The *deformation dimension* ddim X of a compactum X is the minimum n such that any map of X into a CW-complex K is homotopic to one whose image lies in the n-skeleton of K.

The following theorem is a shape version of the classical Whitehead Theorem. Notice that the classical theorem is generalized from connected CW-complexes to connected topological spaces and that the homotopy groups $\Pi_n(X,x)$ are replaced the homotopy pro groups $\Pi_n(\underline{X},\underline{x})$ where $(\underline{X},\underline{x})$ is an inverse system of pointed CW-complexes associated with (X,x).

The Whitehead Theorem in Shape Theory. ([Mos], [Mor], [D_4]).

Let (X,x), (Y,y) *be connected topological spaces,* $n_0 = \max(1 + \dim X, \dim Y) < \infty$ *and* $\underline{f}: (X,x) \to (Y,y)$ *be a shape map such that the induced homomorphism*

$$f_{k\#}: \Pi_k(\underline{X},\underline{x}) \to \Pi_k(\underline{Y},\underline{y})$$

is an isomorphism of pro groups for $1 \leq k < n_0$ *and an epimorphism for* $k = n_0$, *then* $\underline{f}$ *is a shape equivalence.*

Theorem 5.3. *Let* X *and* Y *be continua of finite deformation dimension. If a map* $f:(X,x_0) \to (Y,y_0)$ *induces a pointed shape equivalence* S[f], *then* f *induces a strong shape equivalence* $S^*[f]$.

Proof. Let U be a compact ANR containing X. The natural projection $p: (U,x_0) \to (U \cup_f Y, y_0)$ induces an n-equivalence S[p] for each n, i.e.,

$$\text{pro} - \Pi_k(S[p]): \text{pro} - \Pi_k(U,x_0) \to \text{pro} - \Pi_k(U \cup_f Y, y_0)$$

is an isomorphism for $0 < k < n$ and an epimorphism for $k = n$. Since $Sh(U \cup_f Y) = Sh(U \cup M(f))$ and the latter has finite deformation dimension we may apply the Whitehead Theorem in shape theory and get that p induces a shape equivalence S[p]. Then using Theorem 4.1, we may conclude that f induces a strong shape equivalence $S^*[f]$.

The following theorem is a consequence of Theorem 5.3 and the Fox Theorem in shape theory.

Theorem 5.4. *Let* X *and* Y *be continua of finite deformation dimension. If* $\underline{\underline{f}}: (X,x_0) \to (Y,y_0)$ *is a strong shape morphism such that* $\Theta(\underline{\underline{f}})$ *is an isomorphism, then* $\underline{\underline{f}}$ *is an isomorphism.*

Theorem 5.5. *Let* $f: (X, x_0) \to (Y, y_0)$ *be a map inducing a shape equivalence* S[f], *where* X *and* Y *are movable continua. Then the following conditions are equivalent:*

1) f *induces a strong shape isomorphism* $S^*[f]$,

2) *the pair* (M(f),X) *is movable, where* M(f) *is the mapping cylinder (unreduced) of* $f: X \to Y$.

Proof. 1 → 2 relies on the results of [D_4]. 2 → 1 uses the infinite dimensional Whitehead Theorem in shape theory [D_1] to show S[j] is an isomorphism where the inclusion $j: (U,x_0) \to (U \cup Y, y_0)$ of an ANR U containing X with $U \cap Y = X$ induces an n-equivalence S[j] for each n. Then by Theorem 4.1, $S^*[j]$ is an isomorphism of s-Sh.

Definition. A map $f\colon X \twoheadrightarrow Y$ of compacta is called a CE map (or cell-like map) if $f^{-1}(y)$ is of trivial shape for each $y \in Y$.

Theorem 5.6. *If* $f\colon X \to Y$ *is a* CE *map such that* ddim X, ddim Y $< +\infty$, *then* f *induces a strong shape equivalence* $S^*[f]$.

Proof. Since such a f induces a pointed shape equivalence Theorem 5.3 yields the result.

Definition. [K] A map $f\colon X \to Y$ is called an *hereditary shape equivalence*, provided that for any closed set B in Y the map $(f|A)\colon A \to B$ where $A = f^{-1}(B)$ is a shape equivalence.

Theorem 5.7. *A hereditary shape equivalence of compacta induces a strong shape equivalence.*

In [K] Lemma 8 essentially says that any extension of a hereditary shape equivalence is a shape equivalence. Therefore, Theorem 4.1 implies that a hereditary shape equivalence of compacta induces a strong shape equivalence.

6. Problems

The following questions remain open.

Problem 1. (Compare with [C-S]). Let $\underline{\underline{f}}\colon X \to Y$ be a strong shape morphism such that $\Theta(\underline{\underline{f}})$ is an isomorphism of Sh. Is $\underline{\underline{f}}$ an isomorphism?

Problem 2. Let $\underline{\underline{f}}\colon (X,x) \to (Y,y)$ be a strong shape morphism of pointed continua such that $\Theta(\underline{\underline{f}})$ is an isomorphism. Is $\underline{\underline{f}}$ an isomorphism?

Theorem 6.1. *Let* $\underline{\underline{f}}$: X → Y *be a strong shape morphism of continua such that* $\Theta(\underline{\underline{f}})$ *is an isomorphism. If* (1) ddim X ≤ 1 *or* (2) X *is pointed* 1-*movable and* ddim X *is finite, then* $\underline{\underline{f}}$ *is an isomorphism.*

Proof. By the Fox Theorem in shape theory, we may assume that $\underline{\underline{f}}$: X → Y is induced by the inclusion map i: X → Y. Then S[i] is an isomorphism and using the results of [D_3], the inclusion i induces a pointed shape equivalence i: (X,x) → (Y,x) for each x ∈ X. Then by Theorem 5.4 $S^*[i]$ is an isomorphism of s-Sh.

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University of Washington

Seattle, Washington 98195