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# IRREDUCIBLY ESSENTIAL MAPS FROM INVERSE LIMITS

by

Gary A. Feuerbacher

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**Web:** http://topology.auburn.edu/tp/
**Mail:** Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
**E-mail:** topolog@auburn.edu
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# IRREDUCIBLY ESSENTIAL MAPS FROM INVERSE LIMITS

**Gary A. Feuerbacher**

By a "continuum" is meant a compact, connected metric space. A "polyhedron" is the space of a finite simplicial complex. A "graph" is a connected, one-dimensional polyhedron. A map from a continuum X into a graph G is "essential" if it is not homotopic to a constant map; it is "irreducibly essential" if it is essential, but its restriction to any closed, proper subset of X is inessential.

If T is a continuum, then there is an inverse system $(P_i, F_i^{i+1})_{i \in N}$, with each $P_i$ a polyhedron and each $F_i^{i+1}$ a simplicial map, such that $T = \varprojlim(P_i, F_i^{i+1})$.

In what follows, suppose M is a one-dimensional continuum. Let M be represented as the inverse limit of the inverse system $(X_i, f_i^{i+1})$, with each $X_i$ a graph and each bonding map $f_i^j$ a simplicial map from $X_j$ onto $X_i$. The $i^{th}$ projection map will be denoted $\pi_i$.

*Notation.* In what follows, if Y is a metric space, $d_Y$ will denote its metric. For the factor spaces of the inverse system $(X_i, f_i^{i+1})$, $d_j$ will be used in place of $d_{X_j}$. Whenever each of f and g is a map from a compactum A into a compactum B, $f \cong g$ will mean "f is homotopic to g"; in case $t > 0$, $|f-g| < t$ means that the distance from f to g in the space $B^A$ is less than t, i.e.,

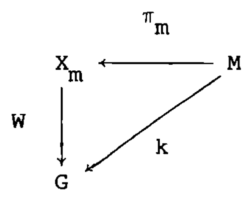
$$\operatorname{lub}_{a \in A}\{d_B(f(a), g(a))\} < t.$$

If Y is a metric space, and $P \in Y$, and $e > 0$, then $B(P,e)$ will denote the open ball with center P and radius e.

The first section of this paper describes certain properties of the inverse system $(X_i, f_i^{i+1})$ which are related to the existence of an essential map from M onto the unit circle $S^1$.

*Lemma* 1. *If* G *is a graph and* k *is a map from* M *into* G, *then there are a positive integer* n *and a map* h *from* $X_n$ *into* G *such that for each* $i \geq 0$, $h \circ f_n^{n+i} \circ \pi_{n+i}$ *is homotopic to* k.

*Proof*. Suppose G is a graph and k a map from M into G. By Theorem 0 of [1, §54, VIII, p. 379], the components of $G^M$ correspond to its homotopy classes, and these components are closed-open. Let $\delta > 0$ be such that the open ball with center k and radius $\delta$ is contained in the component of $G^M$ that contains k. We may regard G as $\varprojlim(Y_i, t_i^{i+1})$ with $Y_i = G$ and $t_i^{i+1} = \mathrm{Id}$ for each i. By Lemma 1 of [2, p. 39], let m be a positive integer and W a map from $X_m$ into G such that the diagram




is $\delta$-commutative. Then $|k - W \circ \pi_m| < \delta$. Also, since $\pi_m = f_m^j \circ \pi_j$ for $j \geq m$, we have

$$|k - W \circ f_m^j \circ \pi_j| < \delta \text{ and}$$

thus $k \cong W \circ f_m^j \circ \pi_j$ for each $j \geq m$.

We have W and m as the needed h and n, respectively.

*Theorem* 1. *If* G *is a graph,* k: M → G *a map,* n *a positive integer, and* h: $X_n \to G$ *a map such that*

$$h \circ f_n^{n+i} \circ \pi_{n+i} \cong k$$

*for each* $i \geq 0$, *then* k *is essential if and only if for each* $i \geq 0$, $h \circ f_n^{n+1}$ *is essential.*

*Proof.* (This argument is a modification of the proof for Q9 in [3, p. 82].

Let G, k, h, and n be as in the Lemma 1. Suppose k is essential. If $i \in N$ and $h \circ f_n^{n+i}$ is inessential, then $(h \circ f_n^{n+i}) \circ \pi_{n+i}$ is inessential, a contradiction.

Now suppose $h \circ f_n^{n+i}$ is essential for each $i \in N$. Suppose k is inessential. Let $t = h \circ \pi_n$; by the Lemma 1, since $t \cong k$, t is inessential. Let $\tilde{G}$ be the universal covering space of G with projection p. Since t is inessential, it may be lifted through $\tilde{G}$; let t* be a lift of t, and let $H = t^*(M)$. Let $\xi$ be an open cover of H by sets open in $\tilde{G}$ such that if $E \in \xi$, then $p|E$ is a homeomorphism from E onto p(E) in G. The Lemma Q3 of [3, p. 80] may be modified to read: For any open cover U of M there exists a positive integer $j > n$ and a finite cover $\nu$ of $X_j$ such that $\{\pi_j^{-1}(V): V \in \nu\}$ refines U. The same argument as given by Case and Chamberlin is valid, substituting "$X_i$" for "B" (representing the figure "8," the union of two circles with a common point). Let U be the collection of all inverse images under t* of elements of $\xi$; $U = \{t^{*-1}(E): E \in \xi\}$. By Q3, let $j > n$ and $\nu$ a finite cover of $X_j$ such that $\{\pi_j^{-1}(V): V \in \nu\}$ refines U.

Let c be the relation, $c \subset X_j \times \tilde{G}$, to which the ordered pair (a,b) belongs if and only if there is a point $Z \in \pi_j^{-1}(a)$

such that $b = t^*(Z)$.

Now, $c$ is a function. For: let $(a,b)$ and $(a,b')$ be in $c$. Let $b = t^*(Z)$ and $b' = t^*(Z')$, with $Z, Z' \in \pi_j^{-1}(a)$. Then $t(Z) = h(\pi_n(Z)) = h(f_n^j(\pi_j(Z))) = h(f_n^j(\pi_j(Z'))) = t(Z')$, hence $p\, t^*(Z) = p\, t^*(Z')$. Let $a \in Q \in \nu$; then since $\{\pi_j^{-1}(V) : V \in \nu\}$ refines $U$, there is $E \in \xi$ such that $t^*(\pi_j^{-1}(a)) \subset E$. But $p$ is one-to-one on $E$; hence since $t^*(Z), t^*(Z') \in E$, and $p\, t^*(Z) = p\, t^*(Z')$, $b' = t^*(Z') = t^*(Z) = b$, and $c$ is single valued.

To show continuity, we note that for $x \in V \in \nu$, and $\pi_j(Z) = x$, $p\, c(x) = p\, t^*(Z) = t(Z) = h(f_n^j(\pi_j(Z))) = h(f_n^j(x))$, and $c(x) = (p|E)^{-1}(h(f_n^j(x)))$ with $E$ an element of $\xi$ such that $t^*(\pi_j^{-1}(V)) \subset E$. Since $h \circ f_n^j$ and $(p|E)^{-1}$ are continuous on $V$, so is $c$. Since $c$ is continuous on each member of $\nu$, $c$ is continuous on $X_j$.

Also for any $x$ in $X_j$, $p\, c(x) = h \circ f_n^j(x)$, i.e., $c$ is a lift of $h \circ f_n^j$ through $\tilde{G}$. Since $G$ is a graph, and $\tilde{G}$ is simply connected, $c(X_j)$ is contractible, and $c$ is inessential. Therefore, $h \circ f_n^j$ is inessential, a contradiction.

*Proposition*. Consider $(S^1, d)$ as a metric space. In what follows, let $\theta > 0$ be such that any two points $a$ and $b$ of $S^1$, with $d(a,b) < \theta$, are non-antipodal. If $H$ is a compactum, and each of $f$ and $g$ is a map from $H$ into $S^1$, with $|f - g| < \theta$, then $f \cong g$ [4, p. 85].

*Definition* 1. Suppose $j$ is a non-negative integer. Suppose $C$ is an infinite sequence of simple closed curves. If, for each $i \geq 1$,

(1) $C_i \subset X_{j+i}$, and

(2) $C_i \subset f_{j+i}^{j+i+1}(C_{i+1})$,

and if (3) there exists a map $h: X_{j+1} \to S^1$ such that, for each positive integer p, $h \circ f_{j+1}^{j+p}|C_p$ is essential, then C will be called an M-cycle. We will say that C is associated with the map h.

If, in addition,

(4) there is a map $k: M \to S^1$ such that $|h \circ \pi_{j+1} - k| < \theta/2$, then C will be called an M-cycle on which k is essential. A *finite* (or infinite) sequence of simple closed curves having properties (1), (2), (3), and (4) will be said to have property p4.

The next result relates the concept of an M-cycle to the notion of a K-cycle, with K being a proper subcontinuum of M; in this case, K is also an inverse limit, i.e.,

$$K = \varprojlim(Y_i, g_i^{i+1}) = \varprojlim(\pi_i(K), f_i^{i+1}|\pi_{i+1}(K)).$$

*Lemma* 2. *If* H *is a subcontinuum of* M, j *is a non-negative integer, and* C *is a sequence of simple closed curves satisfying, for each* i, *(1)* $C_i \subset \pi_{j+i}(H)$, *(2)* $C_i \subset f_{j+1}^{j+i+1}(C_{i+1})$, *and* k *is a map from* M *into* $S^1$, *and* h *is a map from* $X_{j+1}$ *into* $S^1$ *such that* $|h \circ \pi_{j+1} - k| < \theta/2$, *then* C *is an* M-*cycle on which* k *is essential if and only if* C *is an* H-*cycle on which* $k|H$ *is essential.*

*Proof*. Suppose H, j, C, k, and h are as in the hypothesis. Suppose C is an M-cycle on which k is essential. By definition 1, let g be a map, $g: X_{j+1} \to S^1$, such that $|g \circ \pi_{j+1} - k| < \theta/2$, and, for each i, $g \circ f_{j+1}^{j+i}$ is essential on $C_i$. $\left|g \circ \pi_{j+1}|H - k|H\right| < \theta/2$. Also, $H = \varprojlim(\pi_i(H), f_i^{i+1}|\pi_{i+1}(H))$. We have $g \circ \pi_{j+i}|H = (g|\pi_{j+1}(H)) \circ \pi_{j+1}|H$, and C is an H-cycle on which $k|H$ is essential.

Now suppose C is an H-cycle on which $k|H$ is essential. By definition 1, let t be a map, $t\colon \pi_{j+1}(H) \to S^1$, such that $\left|t \circ \pi_{j+1}|H - k|H\right| < \theta/2$, and, for each i, $t \circ f^{j+i}_{j+1}|C_i$ is essential. We have

$$\left|h \circ \pi_{j+1}|H - k|H\right| < \theta/2, \text{ whence}$$
$$\left|t \circ \pi_{j+1}|H - h \circ \pi_{j+1}|H\right| < \theta.$$

This implies that

$$\left|t - h|\pi_{j+1}(H)\right| < \theta, \text{ and, for each } i, \text{ since}$$
$$f^{j+i}_{j+1}(C_i) \subset \pi_{j+1}(H), \ \left|t \circ f^{j+i}_{j+1}|C_i - h \circ f^{j+i}_{j+1}|C_i\right| < \theta.$$

By the proposition, $t \circ f^{j+i}_{j+1}|C_i \cong h \circ f^{j+i}_{j+1}|C_i$, and $h \circ f^{j+i}_{j+1}|C_i$ is essential. Hence, C is an M-cycle on which k is essential.

The next result provides a characterization of essential maps from M into $S^1$ in terms of M-cycles.

*Theorem* 2. *If* k *is a map from* M *into* $S^1$, n *is a positive integer, and* $h\colon X_{n+1} \to S^1$ *is a map such that* $|h \circ \pi_{n+1} - k| < \theta/2$, *then* k *is essential if and only if there is an* M-*cycle* C, *associated with* h, *on which* k *is essential.*

*Proof*. Let k be a map from M into $S^1$. Let n be a positive integer and $h\colon X_{n+1} \to S^1$ a map such that $|h \circ \pi_{n+1} - k| < \theta/2$.

Suppose k is essential. By Theorem 1, $h \circ f^{j}_{n+1}$ is essential for each $j \geq n+1$. Since $X_{n+2}$ is a locally connected continuum, by Theorem 4 of [1, §56, X, p. 430], there is a s.c.c. D contained in $X_{n+2}$ such that $h \circ f^{n+2}_{n+1}|D$ is essential. Let D denote such a s.c.c. Then $h|f^{n+2}_{n+1}(D)$ is also essential. Since the continuous image of a locally connected continuum is locally connected, there is a s.c.c. $E \subset f^{n+2}_{n+1}(D)$ such that

$h|E$ is essential. The sequence $(E,D)$ has property p4. By a similar argument, for each integer $j > 1$, there is a s.c.c. H lying in $X_{n+j}$ such that $h \circ f_{n+1}^{n+j}|H$ is essential, and furthermore, there is a s.c.c. K lying in $f_{n+j-1}^{n+j}(H)$ such that $h \circ f_{n+1}^{n+j-1}|K$ is essential. Now, for each positive integer i, $X_{n+i}$ is a graph, and thus $X_{n+i}$ contains only finitely many s.c.c.s. Therefore, for each positive integer j, the set of all s.c.c.s K lying in $X_{n+j}$ such that $f_{n+1}^{n+j}|K$ is essential is finite. Using an argument analogous to that which shows the existence of an inverse limit on a sequence of finite spaces, each of which has the discrete topology, one deduces the existence of an infinite sequence of s.c.c.s having property p4 (e.g., Theorem 114 of [6]). Hence there is an M-cycle on which k is essential.

Now suppose C is an M-cycle associated with h on which k is essential. Let j be a positive integer. Since $h \circ f_{n+1}^{n+j}|C_j$ is essential, so also is $h \circ f_{n+1}^{n+j}$. By Theorem 1, k is essential.

The second section of this paper describes certain irreducibility properties that the inverse system $(X_i, f_i^{i+1})$ may satisfy. These properties will be related to the notion of an "irreducibly essential" map in the third section.

*Definition* 2. Suppose L is a compact subset of M, n is a positive integer, and C is an M-cycle, with $C_1 \subset X_{n+1}$. Then L is said to be "projection-irreducible about the terms of C" (briefly, "L is irreducible with respect to C") provided that

(1) for each i, $C_i \subset \pi_{n+i}(L)$, and

(2) for each compact, proper subset T of L, there exists $j$ such that $C_j \not\subset \pi_{n+j}(T)$.

*Theorem* 3. *If* n *is a positive integer,* C *is an* M-*cycle,* $C_1 \subset X_{n+1}$, *then there is a compact subset of* M *which is irreducible with respect to* C. *Furthermore, each such point set is connected.*

*Proof.* Let n be a positive integer, C an M-cycle, and $C_1 \subset X_{n+1}$. Let H be the set of all compact subsets K of M such that, for each i, $C_i \subset \pi_{n+i}(K)$. Let H be partially ordered by set inclusion: "$A \leq B$" if and only if $A \subset B$. Let L be a maximal, totally ordered subset of H. Let Y be the common part of all elements of L.

Now, Y is a member of L. For: Let j be a positive integer, and P a point of $C_j$. Let, for each K in L, $g_K = \pi_{n+j}|K$. Suppose $A, B \in L$, and $A \leq B$. Then $g_A = g_B|A$, whence $g_A^{-1}(P) \subset g_B^{-1}(P)$. We have $Q = \{g_K^{-1}(P)\colon K \in L\}$ totally ordered by set inclusion, with $g_A^{-1}(P) \subset g_B^{-1}(P)$ whenever $A \leq B$. Also, each member of Q is a compact point set. Then $\bigcap_{K \in L} g_K^{-1}(P)$ is a compact point set; let R denote it. Since $g_K^{-1}(P) \subset K$, for each K,

$$R \subset Y.$$

We have $C_j \subset \pi_{n+j}(Y)$ for each j, i.e., $Y \in H$. Since $Y \subset K$ for each K in L, and L is maximal, $Y \in L$. Also, since L is maximal, no proper compact subset of Y is in H whence Y is irreducible with respect to C.

Suppose Z is compact and Z is irreducible with respect to C. Suppose Z is not connected. Let $Z = A \cup B$, the sum of 2 mutually exclusive, closed point sets. Let j be a positive integer such that, for each $i \geq 0$, $\pi_{n+j+i}(A)$ does not intersect $\pi_{n+j+i}(B)$.

Since $C_j$ is connected, either $C_j \subset \pi_{n+j}(A)$ or

$$C_j \subset \pi_{n+j}(B);$$

assume $C_j \subset \pi_{n+j}(A)$. Let i be a positive integer. Either $C_{j+i} \subset \pi_{n+j+i}(A)$ or $C_{j+i} \subset \pi_{n+j+i}(B)$; suppose $C_{j+i} \subset \pi_{n+j+i}(B)$. Then $f_{n+j}^{n+j+i}(C_{j+i}) \subset \pi_{n+j}(B)$. But $C_j \subset f_{n+j}^{n+j+i}(C_{j+i})$, a contradiction. We have $C_i \subset \pi_{n+i}(A)$ for each i, and A is a compact, proper subset of Z, whence Z is not irreducible with respect to C, a contradiction.

The next result asserts that we may assume that all M-cycles on which k is essential have their first term in the same factor space and are associated with the same map.

*Lemma* 3. *Suppose* k *is a map from* M *into* $S^1$, *and* D *is an* M-*cycle on which* k *is essential. Let* m *and* n *be non-negative integers,* m < n, *and* s *and* t *be maps from* $X_{m+1}$ *and* $X_{n+1}$ *respectively into* $S^1$, *with* D *associated with* t, *and* s *such that*

$$|s \circ \pi_{m+1} - k| < \theta/2.$$

*Then there is an* M-*cycle* E *associated with* s *such that* $E_{i+n-m} = D_i$ *for each* i. *Furthermore,*

$$|s \circ f_{m+1}^{n+1} - t| < \theta.$$

*Proof*. Let k be a map from M into $S^1$, and D be an M-cycle on which k is essential. Let m < n, s and t be maps from $X_{m+1}$ and $X_{n+1}$, respectively, into $S^1$. Let D be associated with t and let

$$|s \circ \pi_{m+1} - k| < \theta/2. \text{ We have}$$

$$|t \circ \pi_{n+1} - k| < \theta/2, \text{ whence}$$

$$|s \circ \pi_{m+1} - t \circ \pi_{n+1}|$$
$$= |s \circ f_{m+1}^{n+1} \circ \pi_{n+1} - t \circ \pi_{n+1}| < \theta,$$
and $$|s \circ f_{m+1}^{n+1} - t| < \theta.$$

Since t is essential, so are s, $f_{m+1}^{m+2}, f_{m+2}^{m+3}, \cdots, f_n^{n+1}$. The $f_n^{n+1}$-image of $D_1$ is a locally connected subcontinuum of $X_n$. Since $s \circ f_{m+1}^n | f_n^{n+1}(D_1)$ is an essential map onto $S^1$, by Theorem 4 of [1, §56, X, p. 430], there is a simple closed curve L lying in $f_n^{n+1}(D_1)$ such that $s \circ f_{m+1}^n|L$ is essential; let $H_1$ denote such a s.c.c. Similarly, there is a s.c.c. K lying in $f_{n-1}^n$-image of $H_1$ such that $s \circ f_{m+1}^{n-1}|K$ is essential; let $H_2$ denote such a s.c.c. Proceeding by induction, there is a sequence $(H_1, H_2, \cdots, H_{n-m})$ of simple closed curves, with $H_i \subset X_{n+1-i}$, $H_{i+1} \subset f_{n-i}^{n+1-i}(H_i)$, and $s \circ f_{m+1}^{n+1-i}|H_i$ essential for each i. Let E denote the following sequence:

$$E_j = \begin{cases} H_{n+1-m-j} & \text{if } 1 \leq j \leq n-m \\ D_{j-n+m} & \text{if } n-m < j \end{cases}$$

Then E is an M-cycle associated with s on which k is essential.

In the last section we prove the main theorem, which characterizes irreducibly essential maps from M onto $S^1$ in terms of M-cycles and the irreducibility condition discussed in the second section. The final result uses the main theorem to examine hereditary unicoherence in terms of inverse limit properties.

From definitions 1 and 2, and from Theorem 2, we have

*Theorem* 4. *If* k *is a map from* M *onto* $S^1$, *then* k *is irreducibly essential if and only if (1) there is an* M-*cycle on which* k *is essential and (2) if* W *is an* M-*cycle on which* k *is essential, then* M *is irreducible with respect to* W.

*Proof.* Condition (1) is necessary and sufficient for k to be essential. For k to be inessential on every compact proper subset of M, it is necessary and sufficient that k be inessential on every proper subcontinuum of M. Suppose k is irreducibly essential. Let W be an M-cycle on which k is essential. Let H be a proper subcontinuum of M. Then $k|H$ is inessential. By Theorem 2, there is no H-cycle on which $k|H$ is essential. Let j be a non-negative integer such that $W_1 \subset X_{j+1}$. By Lemma 2, if, for each i, $W_i \subset \pi_{j+i}(H)$, then W is an H-cycle on which $k|H$ is essential, a contradiction. Hence M is irreducible with respect to W.

Now suppose condition (2) holds. Let L be a proper subcontinuum of M. Suppose n is a non-negative integer, and h is a map, $h: X_{n+1} \to S^1$, such that $|k - h \circ \pi_{n+1}| < \theta/2$. Suppose $k|L$ is essential. By Theorem 2, let C be an L-cycle on which $k|L$ is essential, with $C_1 \subset \pi_{n+1}(L)$. By Lemma 2, C is an M-cycle on which k is essential. But M is irreducible with respect to C, a contradiction. Thus $k|L$ is inessential, whence k is irreducibly essential.

*Theorem* 5. *If* n *is a positive integer, and* C *is an* M-*cycle*, $C_1 \quad X_{n+1}$, *then* M *is irreducible with respect to* C *if and only if for each positive integer* s, *and each number* $e > 0$, *there is a positive integer* $t > s$ *such that, if* $x \in X_{n+s}$, *then* $d_{n+s}(x, f_{n+s}^{n+t}(C_t)) < e$.

*Proof.* Let n be a positive integer, C an M-cycle, and $C_1 \subset X_{n+1}$. Suppose M is irreducible with respect to C. Let s be a positive integer and $e > 0$. Suppose, by way of contradiction, that for every $t > s$ there is a point $x \in X_{n+s}$

such that $d_{n+s}(x, f_{n+s}^{n+t}(C_t)) \geq e$.

Let W be the following sequence: if i is a positive integer, then

$$W_i = \{x \in X_{n+s}: d_{n+s}(x, f_{n+s}^{n+s+i}(C_{s+i})) \geq e\}.$$

Now, for each i, $W_i$ is closed in $X_{n+s}$. Also since

$$C_{s+i} \subset f_{n+s+i}^{n+s+i+1}(C_{s+i+1}), \text{ and}$$
$$f_{n+s}^{n+s+i}(C_{s+i}) \subset f_{n+s}^{n+s+i+1}(C_{s+i+1}),$$

for each i, we have $W_{i+1} \subset W_i$. Since each term of W is compact, $\bigcap_i W_i$ is a point set; denote it by Y. If $x \in Y$, then for every i,

$$d_{n+s}(x, f_{n+s}^{n+s+i}(C_{s+i})) \geq e.$$

Let $q \in Y$, and let O be the set of all points x such that $d_{n+s}(x,q) < e/2$. By the triangle inequality, if $x \in O$, then $d_{n+s}(x, f_{n+s}^{n+s+i}(C_{s+i})) > e/2$ for every i.

Now, $M - \pi_{n+s}^{-1}(O)$ is a closed, proper subset of M; denote it by M'. Let j be a positive integer. Then

$$f_{n+s}^{n+s+j} \circ \pi_{n+s+j} = \pi_{n+s}, \text{ and}$$
$$f_{n+s}^{n+s+j}(C_{s+j}) \subset X_{n+s} - O = \pi_{n+s}(M')$$
$$= f_{n+s}^{n+s+j}\pi_{n+s+j}(M').$$

Suppose $C_{s+j} \not\subset \pi_{n+s+j}(M')$. Let $p \in C_{s+j}$, but $p \notin \pi_{n+s+j}(M')$. Then there is a point p' in M such that $\pi_{n+s+j}(p') = p$, but $p' \notin M'$. Hence $p' \in \pi_{n+s}^{-1}(O)$. We have $\pi_{n+s}(p') \in O$. Also, $f_{n+s}^{n+s+j}(\pi_{n+s+j}(p')) \in O$, and $f_{n+s}^{n+s+j}(p) \in O$. But $f_{n+s}^{n+s+j}(C_{s+j}) \subset X_{n+s} - O$, a contradiction. Thus $C_{s+j} \subset \pi_{n+s+j}(M')$. Also, for $1 \leq p \leq s$, $C_p \subset f_{n+p}^{n+s}(C_s)$, and $C_s \subset \pi_{n+s}(M')$, whence $C_p \subset \pi_{n+p}(M') = f_{n+p}^{n+s} \circ \pi_{n+s}(M')$. Hence $C_i \subset \pi_{n+i}(M')$ for each i, and M is not irreducible with respect to C, a contradiction.

Now suppose that for each positive integer s and each $e > 0$, there is an integer $t > s$ such that if $x \in X_{n+s}$, then $d_{n+s}(x, f_{n+s}^{n+t}(C_t)) < e$. Suppose M' is a compact, proper subset of M. Let $P \in M - M'$. Let O be a sub-basis element of M, and $P \in O$, and $O \cap M' = \emptyset$. Let q be a positive integer, L an open set in $X_q$, and $O = \pi_q^{-1}(L)$. Then $(f_q^{n+q})^{-1}(L)$ is open in $X_{n+q}$, and $P_{n+q} \in \pi_{n+q}(O) = (f_q^{n+q})^{-1}(L)$, with $\pi_{n+q}(O) \cap \pi_{n+q}(M') = \emptyset$. Let $e > 0$ such that $B(P_{n+q}, e) \subset \pi_{n+q}(O)$. Let t be an integer, $t > q$, such that $d_{n+q}(P_{n+q}, f_{n+q}^{n+t}(C_t)) < e$. If $C_t \subset \pi_{n+t}(M')$, then $f_{n+q}^{n+t}(C_t) \subset \pi_{n+q}(M')$, contradicting $\pi_{n+q}(M') \cap \pi_{n+q}(O) = \emptyset$. Thus M is irreducible with respect to C.

From Theorems 4 and 5, we have immediately

*Theorem* 6. *If* k *is a map from* M *onto* $S^1$, *then* k *is irreducibly essential if and only if (1) there is an* M-*cycle on which* k *is essential, and (2) if* n *is a positive integer, and* C *is an* M-*cycle on which* k *is essential, with* $C_1 \subset X_{n+1}$, *then for each positive integer* s, *and each number* $e > 0$, *there is an integer* $t > s$ *such that, if* $x \in X_{n+s}$, *then* $d_{n+s}(x, f_{n+s}^{n+t}(C_t)) < e$.

*Theorem* 7. *Suppose* H *is a continuum. Then* H *is hereditarily unicoherent if and only if there is no decomposable subcontinuum* H' *of* H *which is the domain space of an irreducibly essential map onto* $S^1$.

*Proof.* Suppose H is hereditarily unicoherent. Let H' be a subcontinuum of H, and g an irreducibly essential map from H' onto $S^1$. Suppose H' is decomposable. Let H' be the sum of two proper subcontinua A and B. Then $g|A$ and $g|B$ are inessential. But H' is unicoherent; thus $A \cap B$ is connected, and g is inessential, a contradiction.

Now suppose each subcontinuum of H which is the initial set of an irreducibly essential map onto the circle is indecomposable. Suppose H is not hereditarily unicoherent. Let K be a subcontinuum of H, and $K = A \cup B$, the sum of two proper subcontinua, such that $A \cap B$ is not connected. Let $A \cap B = C \cup D$, the sum of two mutually exclusive closed sets. Let A' be a subcontinuum of A irreducible between C and D. Let B' be a subcontinuum of B irreducible between $A' \cap C$ and $A' \cap D$. By Urysohn's lemma, let f be a map from A' onto the interval $[0,\frac{1}{2}]$ such that $f(A' \cap C) = 0$, $f(A' \cap D) = \frac{1}{2}$, and the f-image of every other point of A' is in the open interval $\langle 0,\frac{1}{2}\rangle$. Similarly, let g be a map from B' onto $[\frac{1}{2},1]$ such that $g(B' \cap A' \cap D) = \frac{1}{2}$, $g(B' \cap A' \cap C) = 1$, and the g-image of every other point of B' is in the open interval $\langle\frac{1}{2},1\rangle$. Letting $\emptyset$ denote the wrapping function from the real line into the plane, $\emptyset(x) = e^{2\pi ix}$, we define a function h from $A' \cup B'$ into $S^1$, as follows:

$$h(x) = \begin{cases} \emptyset(f(x)) & \text{if } x \in A' \\ \emptyset(g(x)) & \text{if } x \in B' \end{cases}$$

Then h is an irreducibly essential map, whence $A' \cup B'$ is indecomposable, a contradiction.

Using the inverse limit characterization of indecomposability due to D. P. Kuykendall [5], we obtain the following result.

*Corollary. Suppose* M *is the inverse limit of an inverse* *system of one-dimensional polyhedra,* $M = \varprojlim(X_i, f_i^{i+1}, \pi_i)$. *Then* M *is hereditarily unicoherent if and only if the following condition holds: if* H *is a subcontinuum of* M, $H = \varprojlim(\pi_i(H), f_i^{i+1}|\pi_i(H), \pi_i|H) = \varprojlim(Y_i, g_i^{i+1}, \sigma_i)$, n *is a non-negative integer,* t *is a map from* $Y_{n+1}$ *into* $S^1$, *and (1) there is an* H-*cycle* C *associated with* t, *and (2)* H *is irreducible with respect to* D *for each* H-*cycle* D *associated with* t, *then* H *is indecomposable,* i.e., *if* m *is a positive integer, and* $e > 0$, *then there are a positive integer* W *and three points of* $Y_W$ *such that if* P *and* Q *are two of them, and* K *is a subcontinuum of* $Y_W$ *containing* P *and* Q, *then* $d_m(x, g_m^W(K)) < e$, *for each point* x *of* $Y_m$.

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Bellaire, Texas 77401