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**THE PROBLEM OF POINTED VERSUS
UNPOINTED DOMINATION
IN SHAPE THEORY**

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1. Introduction

Let X be a compactum (\cong compact metric space). X is *finitely dominated* if there is a compact polyhedron K and maps $K \begin{smallmatrix} \xrightarrow{u} \\ \xrightarrow{d} \end{smallmatrix} X$ such that $d \circ u$ is homotopic to the identity map of X . X is a *fundamental absolute neighborhood retract*, abbreviated *FANR*, if there exist K as above and shape morphisms $K \begin{smallmatrix} \xrightarrow{u} \\ \xrightarrow{d} \end{smallmatrix} X$ such that $d \circ u$ is the identity shape morphism of X . Thus *FANR*'s play the same role in shape theory as finitely dominated compacta play in homotopy theory.

Now let X be connected, and let $x \in X$. The pointed compactum (X, x) is *pointed finitely dominated* [resp. a *pointed FANR*] if there is a pointed compact polyhedron (K, k) and pointed maps [resp. pointed shape morphisms] $(K, k) \begin{smallmatrix} \xrightarrow{u} \\ \xrightarrow{d} \end{smallmatrix} (X, x)$ such that $d \circ u$ is homotopic to the identity rel. x [resp. $d \circ u$ is the identity morphism in pointed shape theory].

It is well known that X finitely dominated does not imply (X, x) pointed finitely dominated; for example, let X be the comb space [22; p. 26] and let x be the tip of the limit-tooth. However, if x is non-degenerate in X [22; p. 380], then finitely dominated implies pointed finitely dominated. Thus the difference between the two concepts depends only on

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local (more accurately, semi-local) pathology at the base point x .

The corresponding situation in shape theory is quite different. Given (X,x) there is a pointed set called $\varprojlim^1 \text{pro-}\pi_1(X,x)$ obtained by writing $(X,x) = \varprojlim\{(X_n,x_n)\}$ with each X_n a compact polyhedron, and taking $\varprojlim^1\{\pi_1(X_n,x_n)\}$ (\varprojlim^1 of an inverse sequence of groups is defined in [16] or [2] or [15]). When X is connected, the cardinal number of this pointed set, $|\varprojlim^1 \text{pro-}\pi_1(X,x)|$, does not depend on the choice of $\{(X_n,x_n)\}$, nor on the base point x (see §3).

Main Theorem. *Let X be a connected FANR. The obstruction to (X,x) being a pointed FANR is the cardinal number $|\varprojlim^1 \text{pro-}\pi_1(X,x)| - 1$; (X,x) is a pointed FANR iff this number is 0; and the value of this number is independent of the choice of base point.*

Thus the difference between the concepts FANR and pointed FANR is a global matter.

It is conjectured that there exist connected FANR's which are not pointed FANR's. A possible method for constructing an example is suggested in §4, but at present we lack the knowledge of knot groups needed to carry it out. Such an example would be interesting for at least four reasons: (i) in shape theory, it would show that connected FANR's need not be pointed FANR's, and that movable continua need not be pointed movable [1]; (ii) in proper homotopy theory, it would show that a weak proper homotopy equivalence need not be a proper homotopy equivalence (see [4; appendix II] and [12]); (iii) in homotopy theory, it would show that a

homotopy idempotent on a finite complex need not split (see [15]); and (iv) in geometric topology, it would show that an FANR Z -set in the Hilbert Cube need not possess an I -regular neighborhood (see [21]).

This article is intended as a guide to some of the literature on the subject. We rather expect that an example will soon be found of a connected FANR which is not a pointed FANR; perhaps even before these proceedings appear. However, this surprising problem will form a permanent chapter in shape theory - one which those who deal with shape (or proper homotopy theory) must know. So we would regard such an example as the completion rather than the annihilator of the chapter. If, on the other hand, it were proved that no example exists, then the Main Theorem would be vacuous in the shape theory of compacta. It is not vacuous, however, when restated in the more abstract context of pro-homotopy (see §4 or [6]); with slight modification the ideas discussed here would still have interest.

The paper is organized as follows. §2 contains the relevant elementary homotopy theory. This has been known since 1950, but over the years new proofs have appeared in the literature and folklore. We briefly treat all the proofs we know, so that when we come to the shape theory in §3, we can discuss which proofs carry over and which do not. The Main Theorem is proved in §3. §4 contains a short discussion of why we expect that a connected FANR which is not a pointed FANR will be found.

2. The Homotopy Theory

In this section X is a connected compactum; K and L

denote connected but not necessarily compact polyhedra; $x \in X$ is a non-degenerate base point [22; p. 380]; $k \in K$ and $\ell \in L$ (necessarily non-degenerate).

Consider four true statements:

- I. X dominated by $K \Rightarrow X$ homotopy equivalent to some L .
- II. (X, x) dominated by $(K, k) \Rightarrow (X, x)$ homotopy equivalent to some (L, ℓ) .
- III. X homotopy equivalent to $L \Rightarrow (X, x)$ homotopy equivalent to (L, ℓ) .
- IV. X dominated by $K \Rightarrow (X, x)$ dominated by some (L, ℓ) .

We briefly recall some proofs of I - IV.

Proofs of I. (i) Let $p: |SX| \rightarrow X$ be the canonical map from the geometric realization of the singular complex of X . Given $K \xrightarrow[u]{d} X$, let $d': K \rightarrow |SX|$ be a lift of d through p . Then $d' \circ u$ is a homotopy inverse for p . See [24].

(ii) Let $J: X \times I \rightarrow X$ be such that $J_0 = 1_X$ and $J_1 = d_0 \circ u$ where $d_0 \equiv d$. By induction build complexes $K = K_0 \subset K_1 \subset K_2 \subset \dots \subset K_k$ and maps $d_k: K_k \rightarrow X$ and $H^k: (K_k^{k-1} \times I) \cup (K_k \times \{0, 1\}) \rightarrow K_k$ each extending its predecessor, such that $H_0^k = 1$, $H_1^k = u \circ d_k$, and $d_k \circ H^k$ agrees with $J \circ (d_k \times 1)$ where defined. Let $K_\infty = \bigcup_k K_k$, $d_\infty =$ the ultimate extension of d_0 , and H^∞ the ultimate extension of H^0 . Then d_∞ is a homotopy equivalence with homotopy inverse (inclusion) $\circ u$. J and H^∞ provide the required homotopies to the identity maps of X and K_∞ .

(iii) Assume without loss of generality that d is a cofibration. Let $\text{Map}(d \circ u)$ be the "mapping telescope" obtained by gluing together infinitely many copies of the

mapping cylinder of $d \circ u$ end to end. $\text{Map}(d \circ u)$ is homotopy equivalent both to X and to a complex. The details are implicitly in [19].

(iv) A variation on (iii) consists of using the homotopy between $d \circ u$ and l_x to construct a homotopy equivalence from $\text{Map}(u \circ d)$ to X . Since $u \circ d: K \rightarrow K$ is homotopic to a cellular map, $\text{Map}(u \circ d)$ is homotopy equivalent to a complex.

Proofs of II. The above proofs of I all carry over to the pointed case II; we label these (i), (ii), (iii) and (iv). Here are three other proofs:

(v) Given $(K, k) \xrightarrow[u]{d} (X, x)$, attach cells to K to kill the kernels of $d_{\#}$ on homotopy groups. There results a weak homotopy equivalence $d': (L, \ell) \rightarrow (X, x)$. By [23], d' is a homotopy equivalence since (X, x) is dominated by (K, k) .

(vi) Using Brown's Representation Theorem, form a CW approximation (L, ℓ) to (X, x) and appeal to [23] as in (v). See [22; Exercise 7.G.6].

(vii) Let $i: (K, k) \rightarrow (\text{Map}(u \circ d), k)$ be the inclusion map of K into the 0-end of $\text{Map}(u \circ d)$. Then there is a map $j: (\text{Map}(u \circ d), k) \rightarrow (K, k)$ such that $i \circ j$ is a weak homotopy equivalence and $j \circ i = u \circ d$ (see [3, §2]). Then $d \circ j: (\text{Map}(u \circ d), k) \rightarrow (X, x)$ is a weak homotopy equivalence. Compare [17].

Proofs of III: (i) This is a special case of [22; Exercise 7.C.5]. The proof uses standard facts about change of base point.

(ii) Replace the homotopy equivalence $f: L \rightarrow X$ by a fibration $f': L' \rightarrow X$; then X is a retract of L' . Adjoin a

copy of I to L' , by identifying its 0-end with x ; let x' be its 1-end. Then x' is non-degenerate in L' and $(X \coprod_{x=0} I, x')$ is homotopy equivalent both to (X, x) and to (L, ℓ) .

(iii) Similar to (ii), but replace the equivalence $X \rightarrow L$ by a cofibration.

Proofs of IV. IV is a consequence of I and III. More directly, the three proofs of III, above, can all be easily adapted to give direct proofs of IV. We label these (i), (ii) and (iii).

3. The Analogous Shape Theory

We only need the shape theory of compacta and of non-compact polyhedra. A short but adequate account is given on pages 524 and 526-7 of [9]. (The account in [1] is technically not sufficient since non-compact polyhedra are not included.)

In this section X is a connected compactum; K and L denote connected but not necessarily compact polyhedra; $x \in X$, $k \in K$ and $\ell \in L$ are base points. Non-degeneracy of x is *not* needed.

We discuss the analogues of I - IV of §2. Note that since X is compact, X is dominated [resp. shape dominated] by a polyhedron iff X is dominated [resp. shape dominated] by a *compact* polyhedron (see, for example, p. 528 of [9]). Thus to say that X is shape dominated by a complex is to say that X is an *FANR*. Similarly in pointed shape theory.

The analogues of II and III are true:

II': (X, x) shape dominated by $(K, k) \Rightarrow (X, x)$ shape

equivalent to some (L, ℓ) .

III': X shape equivalent to $L \Rightarrow (X, x)$ shape equivalent to (L, ℓ) .

The most obvious analogues of I and IV are conjectured to be *false* (compare [6] where it is shown that the pro-homotopy analogues are certainly false for towers of infinite complexes; see also [8]).

I': X shape dominated by $K \Rightarrow X$ shape equivalent to some L .

IV': X shape dominated by $K \Rightarrow (X, x)$ shape dominated by some (L, ℓ) .

Since II' and III' are true, I' and IV' are equivalent.

Modified analogues of I and IV are true and are stated below as I'' and IV''.

Proofs of II'. First we discuss which of the seven proofs of II given in §2 carry over from homotopy theory to shape theory. Proofs (i) - (iv) do not appear to carry over: in each case one meets a coherence problem among the homotopies involved. Proof (v) carries over: see Theorem 3.1 of [14]. Proof (vi) carries over: see [5] ([5] contains an error which is correctable in the pointed case). Proof (vii) carries over, using the "Whitehead Theorem" 3.2 of [10].

There are at least three other proofs of II' which do not come from homotopy theoretical analogues:

(viii) Write X as $\varprojlim \{X_n\}$ with each X_n a compact polyhedron, and let L be the homotopy inverse limit of $\{X_n\}$; then by [9, as corrected] the natural morphism $(L, \ell) \rightarrow (X, x)$ is a shape equivalence.

(ix) (X, x) has finite shape dimension; and its homotopy pro-groups are stable ([11; Prop. 3.3]). Hence II' follows from Theorem 5.1 of [10].

(x) A more elementary version of (ix) uses Theorem 2.1 of [14] instead of [10].

Proofs of III'. Again we begin with the proofs of III.

(i) does not appear to carry over: change of base point theory does not work in the shape theory of continua which are not pointed 1-movable (definition below), and to assume (X, x) pointed 1-movable is to beg the question, in view of I" and IV" below. (ii) carries over: see [14; Theorem 4.1]. (iii) carries over: dualize the proof of [14; Theorem 4.1].

Other proofs not coming from homotopy analogues are:

(iv) X admits an I-regular neighborhood U , when embedded as a Z -set in the Hilbert Cube [21]. Clearly the inclusion $(X, x) \rightarrow (U, x)$ is a pointed shape equivalence.

(v) In [6] the homotopy theory of [21] is separated from the geometry, yielding a proof of II' more elementary than (iv).

(vi) Write $X = \varinjlim \{X_n\}$ with each X_n a compact polyhedron. By [13; 5.2.9], $\{X_n\}$ is equivalent in $\text{Ho}(\text{tow-CW})$ to a complex, where CW is the category of complexes and maps. Change of base point theory works in $\text{Ho}(\text{tow-CW})$ (though not, as we have said, in $\text{tow-Ho}(\text{CW})$), so II' follows. See [13] for the required definitions.

Remarks on I' and IV'. As we have said, the proofs of I do not appear to carry over. It cannot be expected that proof (i) of IV should carry over, because proof (i) of III

but it also follows from the much more general Theorem 4.1 of [11]: \lim_{\leftarrow}^1 is a functor on pro-Groups. Independence of x follows from Lemma 2.2 of [9] together with the fact that $\lim_{\leftarrow}^1 \text{pro-}\pi_1(X, x)$ is isomorphic to the set of path components of $\text{holim}_{\leftarrow} \{(X_n, x_n)\}$ [2; IX 3.1].

Remarks. (a) The continuous image of a pointed 1-movable continuum is pointed 1-movable [18], [20]. Hence a connected, locally connected FANR is a pointed FANR.

(b) If one can choose $\{(X_n, x_n)\}$ so that the image of $\pi_1(X_m, x_m)$ in $\pi_1(X_n, x_n)$ is a normal subgroup for all n and all $m \geq n$, then it is easy to show that $\{\pi_1(X_n, x_n)\}$ is Mittag-Leffler. Note that this includes the abelian case.

(c) Related matters are discussed in [7] and [8].

(d) Implied throughout this paper is a parallel discussion in weak proper homotopy theory, using the Chapman Complement Theorem [1].

4. Are There FANR's Which Are Not Pointed FANR's?

The strongest evidence for their existence is the example of Dydak and Minc [6], discovered also by Freyd and Heller (unpublished). This is a finitely presented group G and a homomorphism $f: G \rightarrow G$ such that (i) f is not onto, (ii) there exists $g \in G$ such that for all $x \in G$, $f(f(x)) = g \cdot f(x) \cdot g^{-1}$, (iii) $f^2 \neq f$. The inverse sequence $\{X_n\}$, in which each $X_n = K(G, 1)$ and each bond is induced by f , is dominated in pro-homotopy (unpointed!) by $K(G, 1)$; but $\{\pi_1(X_n, x_n)\}$ is easily seen not to be Mittag-Leffler, so $\{(X_n, x_n)\}$ is not dominated in pointed pro-homotopy by a complex. If $K(G, 1)$ were compact, $\lim_{\leftarrow} \{(X_n, x_n)\}$ would be the required FANR. We

understand that Freyd and Heller have shown that G has infinite cohomological dimension, which would imply that this is not the required example.

If G were a knot group, "asphericity of knots" would show that $K(G,1)$ could be a 3-dimensional compact polyhedron, and we would have a 3-dimensional FANR which was not a pointed FANR.

Therefore we ask the

Question. Is there an endomorphism on a knot group having Properties (i) - (iii)?

Weaker question. Is there an endomorphism on a group G with finitely dominated $K(G,1)$ having Properties (i) - (iii)? By a new theorem of S. Ferry [25], $K(G,1)$ would be homotopy equivalent to a compactum Y . The above argument with $X_n = Y$ for all n would give a finite-dimensional connected FANR which is not a pointed FANR, because the inverse sequence $\{X_n\}$ would be "associated with" its inverse limit, in the sense of [26].

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