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## 1. Introduction

A space X is said to be a *Fréchet space* if whenever $x \in \overline{A}$, there exist $x_n \in A$, $n = 1,2,\cdots$, with $x_n \to x$. In general, Fréchet spaces behave very badly with respect to products. In fact, if X and Y are non-discrete Fréchet spaces and $X \times Y$ is Frechet, then a theorem of Michael [5] implies that X and Y must have the following stronger property: if $x \in \bigcap_{n=1}^{\infty} \overline{A}_n$, where $A_1 \supset A_2 \supset \cdots$, then there exists $x_n \in A_n$ with $x_n \to x$. Spaces satisfying this property are called *countably bi-sequential* spaces. We should add that even if X and Y are countably bi-sequential, this does not guarantee that $X \times Y$ is Fréchet (see [4] or [6]).

In a letter to the author, F. Galvin asked the following question: if $X_0, X_1, X_2, \cdots$ are such that $\prod_{i \leq n} X_i$ is Fréchet (equivalently, countably bisequential) for all $n \in \omega$, must $\prod_{i \in \omega} X_i$ be Fréchet (equivalently, countably bi-sequential)? Y. Tanaka [8, Problem 2.6] has asked the same question. In this paper, we construct, assuming Martin's Axiom (MA), a Fréchet space X such that $X^n$ is Fréchet for all $n \in \omega$, but $X^\omega$ is not Fréchet. The space X is countable, and has only one non-isolated point.

Before proceeding with the construction of the example, we would like to mention some related problems. Bi-sequential spaces [5] are closed under countable products, so the space X we construct is a countable countably bi-sequential space which is not bi-sequential. Others (e.g., Galvin [2], Malyhin [4], Olson [6]) have constructed such spaces assuming various axioms of set theory, but no real example has been found. (There are uncountable real examples, e.g., an uncountable $\Sigma$-product of the unit interval.) A space X is called a $\underline{w}$-*space* if whenever $x \in \overline{A}_n$, $n = 1,2,\ldots$, there exists $x_n \in A_n$ with $x_n \to x$. These spaces were introduced by the author in [3], and defined in terms of an infinite game, but this characterization, due to P. L. Sharma [7], is much better. Clearly, every w-space is countably bi-sequential, and the difference between the two classes of spaces does not, on the surface, look very large. But the following question, also asked by Galvin, remains open: if $X^n$ is a w-space for all $n \in \omega$, must $X^\omega$ be a w-space (or a Fréchet space)? A counterexample to this question would be about as far as one could go in this direction. Call X a c*-space (terminology due to Sharma) if X has countable tightness and every countable subset of X is first countable. It is easy to see that if $X^n$ is a c*-space for every $n \in \omega$, then X is a c*-space. No real example of a space which is a w-space but not a c*-space has been found. However, Galvin [1] has constructed such spaces assuming MA.

## 2. Construction of the Example

Unless otherwise stated, we use the letters m, n, and k to denote natural numbers. The example is based on a construction, by induction on the ordinals less than the continuum c, of a certain collection of almost-disjoint subsets of $\omega$. To get us past an uncountable stage $\alpha < c$, we need the following lemma:

*Lemma* (MA). *Let* $\{I_\alpha\}_{\alpha<\kappa}$, $\kappa < c$, *be a collection of infinite almost-disjoint subsets of* $\omega$. *Suppose* $A \subset \omega^n \times \omega^m$, *and* $\{\alpha(0), \alpha(1), \cdots, \alpha(m-1)\} \subset \kappa$ *are such that*

*(1)* $A \subset \omega^n \times \prod_{j<m} I_{\alpha(j)}$

*(2)* $A \cap [(\prod_{i<n} \omega \setminus E(i)) \times (\prod_{j<m} I_{\alpha(j)} \setminus F(j))] \neq \emptyset$ *whenever* $E(i)$ *is a finite union of the* $I_\alpha$*'s, together with a finite subset of* $\omega$, *and* $F(j)$ *is a finite subset of* $\omega$. *Then there is a sequence* $\vec{x}_0, \vec{x}_1, \cdots$ *of elements of* $A$ *such that*

*(i)* $C(\vec{x}_i) \cap C(\vec{x}_j) = \emptyset$ *whenever* $i \neq j$, *where* $C(\vec{x})$ *is the set of coordinates of* $\vec{x}$;

*(ii) if* $\alpha < \kappa$, *then* $I_\alpha \cap \{\pi_i(\vec{x}_j): i < n, j \in \omega\}$ *is finite, where* $\pi_i$ *is the projection on the* $i^{th}$ *coordinate.*

*Proof*. Let $P = \{(f,F): f \subset A, F \subset \kappa$, with $f$ and $F$ finite$\}$. Define $(f,F) < (g,G)$ if and only if

(a) $f \subset g$ and $F \subset G$;

(b) if $\vec{y} \in g \setminus f$, then $\vec{y}$ is an element of $A \cap [(\prod_{i<n} \omega \setminus (\bigcup_{\alpha \in F} I_\alpha) \cup (\bigcup_{\vec{x} \in f} C(\vec{x}))) \times (\prod_{j<m} I_{\alpha(j)} \setminus \bigcup_{\vec{x} \in f} C(\vec{x}))]$.

So defined, $(P,<)$ satisfies the CCC because there are only countably many possible $f$'s, and $(f,F)$ and $(f,G)$ are bounded by $(f, F \cup G)$. For each $\alpha < \kappa$, and $i \in \omega$ let $X_{\alpha,i} = \{(f,F) \in P: |f| > i$ and $\alpha \in F\}$. $X_{\alpha,i}$ is a dense open set in $(P,<)$, so by MA, there is a compatible family $\{(f_{\alpha,i}, F_{\alpha,i}) \in X_{\alpha,i}: \alpha < \kappa, i \in \omega\}$. Pick $\vec{x}_0 \in f_{\alpha(0),i(0)}$. If $\vec{x}_0, \vec{x}_1, \cdots, \vec{x}_{k-1}$ have been chosen, pick $\vec{x}_k \in f_{\alpha(k),i(k)} \setminus \bigcup_{j<k} f_{\alpha(j),i(j)}$. We claim that $\vec{x}_0, \vec{x}_1, \cdots$ is the desired sequence. If $j < k$, then since $\vec{x}_k \in f_{\alpha(k),i(k)} \setminus f_{\alpha(j),i(j)}$, and by the compatibility, the conclusion of property (b) is satisfied with $\vec{y} = \vec{x}_k$ and $f = f_{\alpha(j),i(j)}$. Hence $C(\vec{x}_j) \cap C(\vec{x}_k) = \emptyset$, and so property (i) of the conclusion of the lemma is satisfied. Now let $\alpha < \kappa$. If $\vec{x}_k \notin f_{\alpha,1}$, then the conclusion of (b) is satisfied with $\vec{y} = \vec{x}_k$ and $F = F_{\alpha,1}$. Since $\alpha \in F_{\alpha,1}$, the first n coordinates of $\vec{x}_k$ miss $I_\alpha$. Thus (ii) is satisfied, and this completes the proof.

*Theorem* (MA). *There is a countable Fréchet space* X *such that* $X^n$ *is Fréchet for all* $n \in \omega$, *but* $X^\omega$ *is not Fréchet.*

*Proof.* We will construct a countable space $X_k$ for each $k \in \omega$, so that $\prod_{k \leq n} X_k$ is Fréchet for all $n \in \omega$, but $\prod_{k \in \omega} X_k$ is not Fréchet. We can then take X to be the free union of the $X_k$'s.

To this end, we will construct a sequence $\{\mathcal{I}_n\}_{n \in \omega}$ of collections of infinite subsets of $\omega$ such that $\bigcup_{n \in \omega} \mathcal{I}_n$ is a maximal almost-disjoint collection. We then take $X_k$ to be the space $\omega \cup \{\infty\}$ with the points of $\omega$ isolated, and a neighborhood of $\infty$ is $\omega \cup \{\infty\}$ minus a finite union of elements of $\bigcup_{j \leq k} \mathcal{I}_j$. It is easy to see that, in the space $\prod_{k \in \omega} X_k$, the point $(\infty,\infty,\cdots) \in \mathrm{Cl}\{(n,n,\cdots): n \in \omega\}$, but no sequence of the type $\{(n_k,n_k,\cdots): k \in \omega\}$ converges to $(\infty,\infty,\cdots)$. Thus $\prod_{k \in \omega} X_k$ is not a Fréchet space.

We need to construct the $\mathcal{I}_k$'s so that every finite product of the $X_k$'s is Fréchet. First construct $I_k(n)$, $n \in \omega$, so that $\{I_k(n): n \in \omega, k \in \omega\}$ is an almost-disjoint collection of infinite subsets of $\omega$, with the additional property that for each $k \in \omega$ and finite subset F of $\omega$, there is $n \in \omega$ with $F \subset I_k(n)$.

For each $n \in \omega$, let $\mathcal{A}_n = P(\omega^n)$, and let $\mathcal{A} = \bigcup_{n\in\omega} \mathcal{A}_n$. Let $\mathcal{A} = \{A_\alpha: \alpha < c\}$ so that each element of $\mathcal{A}$ appears c times in the well-ordering. For each $n \in \omega$, define $\beta(n) = n$. Now suppose $I_k(\alpha)$ and $\beta(\alpha)$ have been defined for all $\alpha < \kappa$, where $\omega \leq \kappa < c$, and $k \in \omega$. Let $\mathcal{I}(\kappa) = \{I_k(\alpha): \alpha < \kappa, k \in \omega\}$. Let $\beta(\kappa)$ be the least ordinal $\beta$ such that $\beta > \beta(\alpha)$ whenever $\omega \leq \alpha < \kappa$, and such that $A_\beta \subset \omega^n$ satisfies the following two properties:

(i) there are a set $J \subset \{0,1,\cdots,n-1\} = n$, and $\{I_j: j \in J\} \subset \mathcal{I}(\kappa)$ so that $A_\beta \subset (\prod_{i\in n\setminus J} \omega) \times (\prod_{j\in J} I_j)$;

(ii) $A_\beta \cap [(\prod_{i\in n\setminus J} \omega\setminus E(i)) \times (\prod_{j\in J} I_j\setminus F(j))] \neq \emptyset$ whenever $E(i)$ is a finite union of elements of $\mathcal{I}(\kappa)$, and $F(j)$ is a finite subset of $\omega$.

Note that n is uniquely determined by $A_\beta$, but the set J depends also on $\kappa$. Also, such a $\beta$ always exists since $\omega$ itself, with $n = 1$ and $J = \emptyset$, satisfies (i) and (ii).

By the lemma, there is a sequence $\vec{x}_0, \vec{x}_1, \cdots$ in $A_{\beta(\kappa)}$ such that $C(\vec{x}_i) \cap C(\vec{x}_j) = \emptyset$ for $i \neq j$, and $I \cap \{\pi_i(\vec{x}_k): k \in \omega, i \in n\setminus J\}$ is finite whenever $I \in \mathcal{I}(\kappa)$. Express $\omega$ as $\bigcup_{m\in\omega} W_m$, where $W_m$ is infinite and $W_m \cap W_{m'} = \emptyset$ if $m \neq m'$. Define $I_m(\kappa) = \{\pi_i(\vec{x}_k): k \in W_m, i \in n\setminus J\}$. The inductive step is now complete.

Let $\mathcal{I}_k = \{I_k(\alpha): \alpha < c\}$, and let $X_k$ be as defined earlier. We have already shown that $\prod_{k\in\omega} X_k$ is not Fréchet. It remains to prove that $\prod_{k<n} X_k$ is Fréchet for each $n \in \omega$. To this end, suppose $A \subset \prod_{k<n} X_k$, and $x \in \overline{A}\setminus A$. We need to show there exists $x_n \in A$ with $x_n \to x$. We will prove this for the case $A \subset \omega^n$ and $x = (\infty,\infty,\cdots,\infty) = \infty^n$, the other cases being trivial or reducible to a case similar to this one.

Let $\mathcal{I} = \bigcup_{n\in\omega} \mathcal{I}_n$. Suppose $A \cap (\prod_{i<n} \omega\setminus E(i)) = \emptyset$, where $E(i)$ is a finite union of elements of $\mathcal{I}$. Then $A \subset \bigcup_{i<n} (\omega \times \cdots \times \omega \times E(i) \times \omega \times \cdots \times \omega)$, so there exists $j(0) < n$ and $I_{j(0)} \in \mathcal{I}$ so that $I_{j(0)} \subset E(j(0))$, and $\infty^n \in Cl(A(0))$, where $A(0) = A \cap [\omega \times \cdots \times \omega \times I_{j(0)} \times \omega \times \cdots \times \omega]$. Now suppose $A(0) \cap [(\Pi\{\omega\setminus E(i)': i \in n\setminus\{j(0)\}\}) \times (I_{j(0)}\setminus D(j(0)))] = \emptyset$, where $E(i)'$ is a finite union of elements of $\mathcal{I}$ and $D(j(0))$ is a finite subset of $\omega$. (We are using the subscript to indicate position in the product, in order to simplify notation.) Then there exists $j(1) \in n\setminus\{j(0)\}$ so that $\infty^n \in Cl(A(1))$, where $A(1) = A(0) \cap [\omega \times \cdots \times \omega \times I_{j(1)} \times \omega \times \cdots \times \omega \times I_{j(0)} \times \omega \times \cdots \times \omega] = A(0) \cap \Pi\{\omega: i \in n\setminus\{j(0), j(1)\}\} \times I_{j(0)} \times I_{j(1)}$. We continue the process until we have a set $J = \{j(0),\cdots,j(m)\}$ and $A(m) \subset (\prod_{i\in n\setminus J} \omega) \times \prod_{j\in J} I_j$ with $\infty^n \in Cl(A(m))$ and $A(m) \cap [(\prod_{i\in n\setminus J} \omega\setminus E(i)) \times (\prod_{j\in J} I_j\setminus F(j))] \neq \emptyset$ whenever $E(i)$ is a finite union of elements of $\mathcal{I}$ and $F(j)$ is a finite subset of $\omega$.

Choose $\kappa_0$ large enough so that $\{I_j: j \in J\} \subset \mathcal{I}(\kappa_0)$. Now $A(m) = A_\beta$ for $c$ $\beta$'s, so choose $\beta_0 > \sup\{\beta(\alpha): \alpha < \kappa_0\}$ such that $A(m) = A_{\beta_0}$. Then for any $\kappa_0 \leq \kappa < c$, it is true that $A_{\beta_0}$, $J$, and $\kappa$ satisfy (i) and (ii) in the above construction of the $\mathcal{I}_k$'s. Thus $\beta_0 = \beta(\kappa)$ for some $\kappa_0 \leq \kappa < c$, and we have the sequence $\vec{x}_0, \vec{x}_1, \cdots$ in $A_{\beta(\kappa)}$ that we chose in the construction. It is easy to see from the definition of $X_i$ that the set $\{\pi_i(\vec{x}_k): k \in W_n\}$ converges to $\infty$ in $X_i$ for each $i < n$, and since $C(\vec{x}_j) \cap C(\vec{x}_k) = \emptyset$ for $j \neq k$, then $\{\vec{x}_k: k \in W_n\}$ converges to $\infty^n$. This completes the proof.

*Remark.* We can get an example with only one non-isolated point as follows: let Y be the space which is the free union X of the $X_k$'s, with the points "$\infty$" identified to a single point $\hat{\infty}$. Let $\pi: X \to Y$ be the projection. Define a neighborhood of $\hat{\infty}$ to be of the form $\pi(U_1 \cup \cdots \cup U_n \cup X_{n+1} \cup X_{n+2} \cup \cdots)$, where $U_i$ is an open set in $X_i$ containing $\infty$.

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