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by

PAUL R. PATTEN

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Web: <http://topology.auburn.edu/tp/>

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

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LOCALLY CONNECTED GENERALIZED ABSOLUTE NEIGHBORHOOD RETRACTS¹

Paul R. Patten

In 1952 Hiroshi Noguchi [No, p. 20] defined the following generalization of absolute neighborhood retracts.

Definition. A compact subset X of a metric space Y is an *approximative neighborhood retract of Y in the sense of Noguchi* if there is a neighborhood U of X in Y such that for every positive real number ϵ there is a map r from U into X such that $d(r|_X, id_X) < \epsilon$ [Gm1, p. 61], [Gm2, p. 9], [Gr, pp. 16-17], [Fi, p. 42].

Definition. A compact metric space X is an *approximative absolute neighborhood retract in the sense of Noguchi* ($AANR_N$) provided it is an approximative neighborhood retract in the sense of Noguchi of any metric space in which it can be embedded.

A further generalization which allows the neighborhood to vary with ϵ as well as the map r was given in 1970 by Michael Clapp. These spaces are called *approximative absolute neighborhood retracts in the sense of Clapp* ($AANR_C$) [Cl, p. 118], [Fi, p. 43].

The importance of these generalizations is that they satisfy certain fixed point properties which are well known for the case of ANR's. Such properties are as follows:

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- (1) Every null-homotopic map of an AANR_C into itself has a fixed point [Cl, Theorem 5.7, p. 126].
- (2) Every acyclic AANR_C has the fixed point property [Cl, Theorem 5.6, p. 126].

In some sense, however, these generalizations of compact ANR's are very far from the concept of an ANR. In particular, if one wishes to study the properties of the image of a compact ANR under a certain class of maps it is clear that the image must at least be locally connected. Since there are, nevertheless, trivial examples of AANR 's which are not locally connected, it may be profitable to study generalized ANR 's that must be locally connected.

The first step in this direction is a modification of the definitions given above.

Definition. By a *surjective approximative retract* of a metric space Y will be meant a compact subspace X of Y such that for every positive real number ϵ there is a map r from Y onto X such that $r(X) = X$ and $d(r|_X, \text{id}_X) < \epsilon$.

Definition. By a *surjective approximate neighborhood retract* of a metric space Y in the sense of *Noguchi* will be meant a compact subspace X which is a surjective approximate retract of some neighborhood of X in Y .

Definition. By a *surjective approximate neighborhood retract* of a metric space Y in the sense of *Clapp* will be meant a compact subspace X of Y such that for every positive real number ϵ there are a neighborhood U of X in Y and a map r from U onto X such that $r(X) = X$ and $d(r|_X, \text{id}_X) < \epsilon$.

If a compact metric space is a surjective approximative retract (a surjective approximative neighborhood retract in the sense of Noguchi or Clapp) of every metric space in which it can be embedded, then it is said to be a *surjective approximative absolute retract* (a *surjective approximative absolute neighborhood retract in the sense of Noguchi or Clapp*).

These will be abbreviated as SAAR, SAANR_N, or SAANR_C. It is clear that since any SAAR, SAANR_N, SAANR_C X is assumed to be compact, it may be embedded in the Hilbert cube Q. Since any neighborhood of X in the locally connected space Q contains a compact locally connected neighborhood of X, it is clear that X is the image of a compact locally connected space under an identification. Hence, X is locally connected. This means that the class of SAAR, SAANR_N, or SAANR_C is properly contained in the class of AAR, AANR_N, or AANR_C respectively. Thus, the fixed point properties (1) and (2) still hold for these new generalizations. Of course, it is important to ask that the surjective approximate retracts have certain properties in common with ANR's which the more general versions fail to have. In this direction an apparently different type of generalization of ANR will be made.

Definition. A compact metric space X is said to be a *quasi-ANR* (q-ANR) if for every positive real number ϵ there exist an ANR A, a map f from X onto A, and a map g from A onto X such that gf is ϵ -near id_X (i.e. $d(gf(x), x) < \epsilon$ for all x in X). If the choice of A is independent of ϵ then the space X will be called a *uniform q-ANR*.

If in the above definition the ANR's can be replaced

by AR's the space will be called a *quasi-AR* (q-AR).

The surprising property which the q-ANR's satisfy is contained in the following lemma which has been proved in the ANR-case by Lončar and Mardešić [Lo&Ma, Lemma 1].

Lemma 1. *Suppose A is a q-ANR and f is a component preserving map of a compact metric space X onto A. Then given a positive real number ϵ there is a positive real number δ such that if g is a δ -map from X onto a metric space Y, then there is a map h from Y onto A which satisfies $d(hg, f) < \epsilon$.*

Proof. We will use the version of this lemma where A is an ANR. This is the version which has been proved in Lončar and Mardešić.

Let $\epsilon > 0$ be given. Since A is a q-ANR there are a compact ANR B, a map ϕ from A onto B, and a map ψ from B onto A such that $\psi\phi$ is $\epsilon/2$ -near id_A . Since ψ is uniformly continuous there is a positive real number η such that if b_1 and b_2 belong to B with $d(b_1, b_2) < \eta$ then $d(\psi(b_1), \psi(b_2)) < \epsilon/2$. By making sure that $\psi\phi$ is nearer to the identity on A than half the minimum distance between distinct components of A, one can guarantee that ϕ preserves components. Thus, ϕf from X onto B may be assumed to be component preserving.

Since B is an ANR and ϕf is component preserving we may apply the lemma for the ANR case and obtain a δ corresponding to η . Now suppose g is a δ -map from X onto a metric space Y. By the ANR case there is a map Π from Y onto B such that Πg is η -near ϕf . Hence, $\psi\Pi g$ is a map from Y onto A such that $\psi\Pi g$ is $\epsilon/2$ -near $\psi\phi f$, which in turn is $\epsilon/2$ -near f. Hence by

letting $h = \psi \Pi$ we have a map from Y onto A satisfying the conclusion of the lemma.

It is easy to see that this lemma is another characterization of a q -ANR. Suppose A is a compact metric space which makes lemma 1 true. That is, if f from a compact metric space X onto A is component preserving, then given $\epsilon > 0$ there is a $\delta > 0$ such that for any δ -map of X onto a metric space Y there is a map h from Y onto A which satisfies $d(hg, f) < \epsilon$. Then it is claimed that A must be a q -ANR. To see this it is necessary only to observe that the map id_A from A onto A is component preserving, and that for every $\delta > 0$ there is a δ -map from A onto some finite polyhedron B which is also an ANR. Thus, results about polyhedra and ANR's which depend on lemma 1 are results which hold for q -ANR's. See [Lo&Ma, Theorem 1] and [Fo&Rg, section 3].

The connection between q -ANR's and SAANR_C 's will be given in the following lemmas and theorem.

Lemma 2. If X is a surjective approximative neighborhood retract of the Hilbert cube Q in the sense of Clapp (Noguchi) then it is an SAANR_C (SAANR_N) [Cl], [Fi].

Since the proof of this lemma is similar to the corresponding proof for the other generalized ANR's it will be omitted.

Lemma 4. Every q -ANR is a SAANR_C .

Proof. It may be assumed that the q -ANR X is contained in Q . Let $\epsilon > 0$ be given. Since X is a q -ANR there exist an ANR A , a map f from X onto A , and a map g from A onto X

such that gf is ϵ -near id_X . Since A is also a neighborhood extensor there is a map ϕ into A which extends f to some neighborhood U of X in Q . Now let $r = g\phi$. Then r is a map from U onto X such that $r(X) = g\phi(X) = gf(X) = X$ and $d(r|_X, \text{id}_X) = d(gf, \text{id}_X) < \epsilon$. Thus, X is a surjective approximative neighborhood retract of Q in the sense of Clapp. Therefore, by lemma 2 X is a SAANR_C .

The surprising thing about lemma 3 is that the converse is also valid. To see that, we need the following lemmas.

Lemma 4. Let X be a SAANR_C . Then for every positive real number ϵ there is a positive real number δ such that if f from X onto some metric space Y is a δ -map, then there is a map g from Y into X so that gf is ϵ -near id_X [No, p. 21].

Proof. Notice that the proof given in Noguchi is still valid if the neighborhoods as well as the maps are allowed to vary with ϵ .

Lemma 5. Let Q be the product of intervals $[0, 1/i]$ with the restriction of the Hilbert space metric as a metric. Suppose X is a locally connected compact metric space embedded in Q . Then for every $\eta > 0$ there is a map h from X into Q such that

- (i) if C is a non-degenerate component of X , then $h(C)$ contains $N_\eta(C)$, and*
- (ii) h is 3η -near the embedding of X [Lo&Ma, lemma 5, p. 42].*

Proof. The proof for this lemma may be found in the Lončar and Mardešić paper.

Theorem 1. A compact metric space is a q -ANR if and only if it is a $SAANR_C$.

Proof. One way has already been proven in lemma 3.

Assume that X is a $SAANR_C$ contained in $Q = \Pi[0,1/i]$.

Let $\varepsilon > 0$ be given. Since X is a $SAANR_C$ there are a neighborhood $N_{4\eta}(X)$ of X in Q and a map r from $N_{4\eta}(X)$ onto X such that $r(X) = X$ and $d(r|_X, id_X) < \varepsilon$. Notice that $\eta > 0$ can be chosen so that if $d(a_1, a_2) < 4\eta$ for elements in $N_{4\eta}(X)$, then $d(r(a_1), r(a_2)) < \varepsilon$. By lemma 4 there is a $\delta > 0$ such that if f from X into a metric space Y is a δ -map, then there is a map g from Y into X such that gf is η -near id_X .

Since X is compact there is a δ -map f from X onto some polyhedron Y which is also an ANR. Hence, there is a map g from Y into X such that $d(gf(x), x) < \eta$ for all x in X . Hence X is contained in $N_\eta(g(Y))$. Clearly η may be chosen so that gf is component preserving. Thus, on the degenerate point components $\{x\}$ of X $gf(x) = x$. According to lemma 5 there is a map h from $g(Y)$ into Q such that $hg(Y) \supset X$, h fixes the point components of $g(Y)$, and $d(hg(y), g(y)) < 3\eta$ for all y in Y . It is clear that $d(hgf(x), x) < 4\eta$ for all x in X . Thus, $rhgf$ is a map from X into X . Now $X \supset rhgf(X) = rhg(Y) \supset r(X) = X$ and $d(rhgf(x), x) < 2\varepsilon$ for all x in X . Therefore, rhg is a map from Y onto X such that $rhgf$ is 2ε -near id_X . Hence, X is a q -ANR.

In the following some results about q -ANR's, q -AR's, and SAAR's will be given.

Definition. A compact metric space will be said to be *quasi-contractible* provided that for each $\varepsilon > 0$ there is a

homotopy F_t of X into itself so that F_0 is a constant map, $F_1(X) = X$, and F_1 is ϵ -near id_X .

Lemma 6. A quasi-contractible SAANR_C is a SAAR [Cl, p. 128].

Proof. It suffices to show that if the quasi-contractible SAANR_C X is embedded in Q , then it is a surjective approximate retract of Q .

Let $\epsilon > 0$ be given. Since X in Q is quasi-contractible there is a homotopy F_t such that F_0 is constant, $F_1(X) = X$, and $d(F_1(x), x) < \epsilon/2$ for all x in X . Now since X is a SAANR_C there exist a neighborhood U of X in Q and a map r from U onto X such that $r(X) = X$ and $d(r|_X, \text{id}_X) < \epsilon/2$. It is clear that F_0 extends to a map of Q into U (again a constant map).

We now make use of the following lemma found in Borsuk [Bk, p. 94]. *Let X be a closed subset of a metrizable space Z . Let U be a metric ANR. Let F_t be a homotopy from X into U so that F_0 has an extension to all of Z . Then there is a homotopy H_t from Z into U such that H_t extends F_t for all t .* In particular, for our neighborhood U found above there is an extension H_1 of F_1 to all of Q . Consider rH_1 from Q into X . Then $rH_1(X) = rF_1(X) = r(X) = X$ and $d(rH_1|_X, \text{id}_X) \leq d(rF_1, F_1) + d(F_1, \text{id}_X) < \epsilon$. Hence, X is a surjective approximate retract of Q .

By embedding a SAAR X in Q and using the fact that Q is contractible the converse of lemma 6 follows. Thus, we have the following theorem.

Theorem 2. A compact metric space is a SAAR if and only

if it is a quasi-contractible $SAANR_C$.

By following the proof of lemma 3 it can be seen that every q -AR is a SAAR. This leads naturally to the question; is every SAAR a q -AR? A partial answer is the following theorem.

Theorem 3. Every AR-like SAAR is a q -AR.

Proof. Let X be an AR-like SAAR. Then clearly X is a $SAANR_C$; hence, X is a q -ANR. Let $\epsilon > 0$ be given. Then by lemma 1 there is a $\delta > 0$ such that if g is a δ -map from X onto a metric space Y , then there is a map h from Y onto X such that hg is ϵ -near id_X . Since X is AR-like there is a δ -map from X onto an AR A . Hence, there is an h from A onto X such that hg is ϵ -near id_X . This proves that X is a q -AR.

The Hawaiian earring H (figure 1) is an example of a q -ANR which is not an ANR. The Hawaiian earring also fails to be a $SAANR_N$; hence, it can not be either a q -AR or SAAR, nor can it be quasi-contractible. It is interesting to observe that the cone over H , $C(H)$, is a contractible (hence, quasi-contractible) q -ANR. Furthermore, since $C(f)$ from $C(H)$ onto $C(A)$ is an ϵ -map whenever f from H onto A is, $C(H)$ is an AR-like SAAR; thus, by theorem 3 $C(H)$ is a q -AR. Since like H , $C(H)$ fails to be locally contractible, $C(H)$ is not an AR.

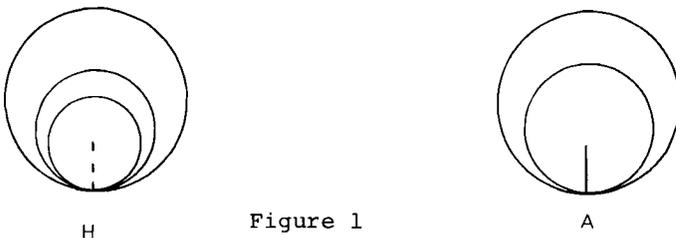


Figure 1

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University of Oklahoma

Norman, Oklahoma 73069

Dept. of Math & Science

Brewton Parker College

Mount Vernon, Georgia 30445