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ALMOST EVERYTHING YOU WANTED TO KNOW ABOUT HOMOGENEOUS, CIRCLE-LIKE CONTINUA

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A continuum is a compact, connected, nonvoid metric space. A continuum X is homogeneous if for each pair of points x and y in X, there exists a homeomorphism of X onto itself that maps x to y. The simple closed curve is the only obvious homogeneous plane continuum.

In an effort to construct other homogeneous plane continua, one might consider the chainable continuum pictured in Figure 1 and formed from the product of the Cantor set and the interval [0,1] by adding horizontal line segments. The first horizontal line segment joins (1/3,0) and (2/3,0); the second and third join the points (1/9,1) and (2/9,1) and the points (7/9,1) and (8/9,1), respectively. The horizontal line segments continue to alternate between the top and bottom according to the length of the gaps of the Cantor set. Call this continuum X_1.

The continuum X_1 is not homogeneous; for one thing X_1 is locally connected at some, but not all, of its points. To ameliorate this, consider the chainable continuum X_2 pictured in Figure 2 and obtained from X_1 by shrinking the horizontal line segments to points.

The continuum X_2 is also not homogeneous; for instance, X_2 contains exactly four endpoints (A point x of a chainable continuum X is an endpoint if X can be chained by chains with arbitrarily small mesh so that x is always in the first link). To avoid this problem, we must either make every point an endpoint or eliminate these four endpoints. Choose the latter alternative, and glue together the two end intervals to get the circle-like continuum X_3 pictured in Figure 3.

The continuum X_3 is rich in homeomorphisms, namely radial shifts along the arcs and rotations. However, X_3 is not homogeneous since it contains arcs, which have endpoints and nonendpoints, local separating points and non-local separating points. Note parenthetically that the quotient space of the monotone decomposition of X_3 into arc components is a simple closed curve. According to a theorem of F. B. Jones [7], the arcs must be replaced by homogeneous continua.

We recall that R. H. Bing [1,3] has proved that the pseudo-arc is the only homogeneous, chainable continuum (every point of the pseudo-arc is an endpoint [2]). Hence, if pseudo-arcs are intricately meshed together in place of the arcs of X_3, possibilities are good for a homogeneous continuum. (Caution: it helps to take weaving lessons from either Bing or Jones.)

Let us define a circle of pseudo-arcs to be a circle-like continuum that has a continuous decomposition into pseudo-arcs such that the decomposition space is a circle. R. H. Bing and F. B. Jones [4] have proved the following:

Theorem 1. *The circle of pseudo-arcs exists. It is topologically unique and homogeneous.*

C. E. Burgess [5] and Jones [8] have characterized the planar circle-like homogeneous continua.

Theorem 2. *The homogeneous circle-like plane continua are precisely the circle, the pseudo-arc, and the circle of pseudo-arcs.*

A point (which, after all, is a homogeneous continuum) is not a circle-like continuum. We might view, however, the pseudo-arc and the circle of pseudo-arcs as being constructed by blowing up every point of the point or the circle, respectively, into pseudo-arcs. An apostle of this viewpoint might scurry about, looking for homogeneous continua so that he could swell their points to pseudo-arcs. Coming across a solenoid, he would define a solenoid of pseudo-arcs to be a circle-like continuum with a continuous decomposition to a solenoid such that each decomposition element is a pseudo-arc. We recall that solenoids are the circle-like continua that are inverse limits of circles with bonding maps being covering maps. Except for the circle (which is sometimes not included in the class of solenoids), all solenoids are nonplanar. Solenoids are homogeneous; in fact, they are topological groups.

Recently, two theorems about solenoids of pseudo-arcs have appeared. The first [9] is the author's existence theorem.

Theorem 3. *Given a solenoid* S, *there exists a solenoid of pseudo-arcs* M *such that* M *decomposes to* S. *Furthermore* M *is homogeneous.*

The second theorem is a classification theorem due to C. L. Hagopian and the author [6].

Theorem 4. *Each homogeneous, nonplanar, circle-like continuum is either a solenoid or a solenoid of pseudo-arcs.*

A question about homogeneous, circle-like continua remains open.

Uniqueness Question. Suppose M *and* N *are solenoids of pseudo-arcs that decompose to the same solenoid. Are* M *and* N *homeomorphic?*

If someone gives a negative answer to this uniqueness question, then more work remains to be done on homogeneous, circle-like continua. If someone gives a positive answer, however, then he can address the next conference under the title *Everything you wanted to know about homogeneous, circle-like continua.*

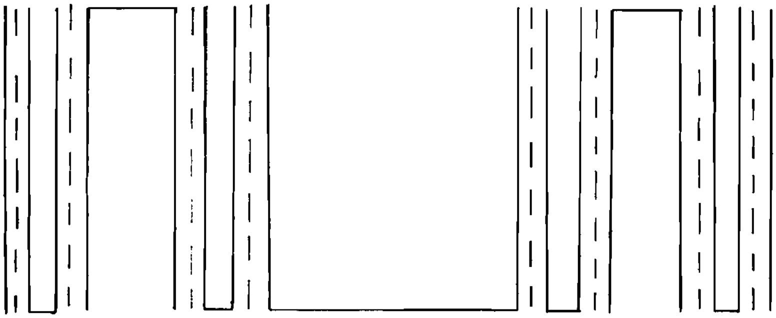

Figure 1

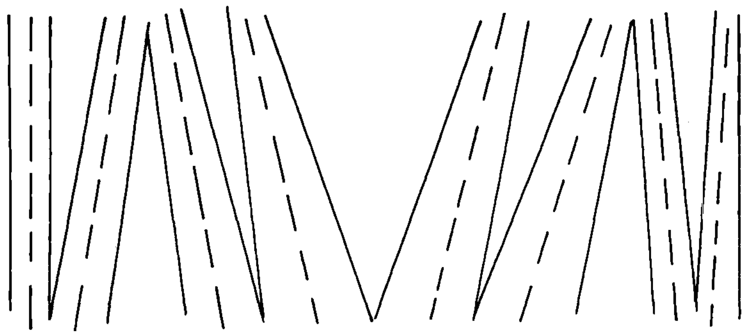

Figure 2

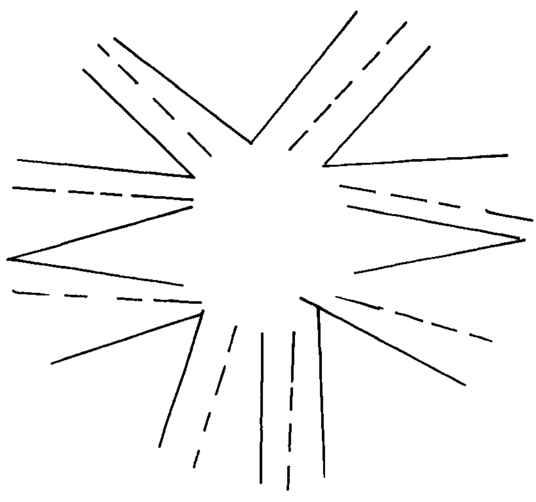

Figure 3

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