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Let X denote a Hausdorff space and let c(X) denote the set of all compact subsets of X. A compact cover F of X is said to be CS-*cofinal* [R-S] if there is a function g: c(X) → F satisfying:

(1) if A ∈ c(X) then A ⊂ g(A), and

(2) if A,B ∈ c(X) and A ⊂ B, then g(A) ⊂ g(B).

The concept of CS-cofinal is used to help reduce the set of compact subsets determining the compactly generated shape of a space. The function g: c(X) → F is called a CS-cofinality function for F.

A compact cover F of X that is CS-cofinal is said to be CS-*finite* if for each A ∈ F there are only finitely many B ∈ F such that B ⊂ A. The Hausdorff space X is said to be CS-*finite* if there is a compact cover F of X that is CS-finite. From Example 4.5 of [R-S], every paracompact, locally compact Hausdorff space is CS-finite. Using these definitions, Example 4.9 and Corollary 4.10 of [S] may be restated as follows:

(1) *Proposition. If two* CS-*finite metric spaces have the same Borsuk-strong shape* [B-2], *then they have the same compactly generated shape* [R-S].

(2) *Corollary. If two locally compact metric spaces have the same Borsuk-strong shape, then they have the same compactly generated shape.*

A question that arises is when does (1) apply and (2) not apply? That is, are there metric spaces that are CS-finite and not locally compact?

(3) *Proposition. If* X *is a Hausdorff space that fails to be locally compact at a point* x_0 *at which* X *has a countable local base, then* X *is not* CS *finite.*

The following proof of the proposition is an adaptation of a similar construction given by W. L. Young for the case $X = (0,1] \times [-1,1] \cup \{(0,0)\}$.

Proof of (3). Let U_n be a countable local base of X at the point x_0. Assume without loss that $U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots$, and that each inclusion is proper. Since X fails to be locally compact at x_0, for all n, $\overline{U}_n$ is not compact.

Let F be any compact cover of X that is CS-cofinal and let $g\colon c(X) \to F$ be a CS-cofinality function for F. There is a sequence $\{x_n\}$ that converges to x_0 such that, for all n,

$$x_{n+1} \in U_{n+1} \setminus g(S_n) \quad \text{where} \quad S_n = \{x_1, x_2, \cdots, x_n\}.$$

Then $\{g(S_n)\}$ is a sequence of sets in F such that, for all n, $g(S_n)$ is a proper subset of $g(S_{n+1})$. But $S = \cup_1^\infty S_n \cup \{x_0\}$ is a compact set and for all n, $g(S_n) \subset g(S)$. Thus X cannot be CS-finite.

(4) *Corollary. For metric spaces, the concepts of locally compact and* CS-*finite are equivalent.*

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