

---

# TOPOLOGY PROCEEDINGS



Volume 3, 1978

Pages 199–204

---

<http://topology.auburn.edu/tp/>

## UNORDERED TYPES OF ULTRAFILTERS

by

S. SHELAH AND M. E. RUDIN

---

### Topology Proceedings

**Web:** <http://topology.auburn.edu/tp/>

**Mail:** Topology Proceedings  
Department of Mathematics & Statistics  
Auburn University, Alabama 36849, USA

**E-mail:** [topolog@auburn.edu](mailto:topolog@auburn.edu)

**ISSN:** 0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

## UNORDERED TYPES OF ULTRAFILTERS

**S. Shelah and M. E. Rudin**

Suppose that  $\kappa$  is a cardinal. If  $U$  and  $V$  are ultrafilters on  $\kappa$  and  $f: \kappa \rightarrow \kappa$  is a function, we say that  $f(U) = V$  if  $\{f(H) \mid H \in U\} = V$ . We say that  $V \leq U$  if there exists an  $f$  with  $f(U) = V$ . We say that  $U$  and  $V$  are of *the same type* (or  $U = V$ ) if both  $V \leq U$  and  $U \leq V$ : This is an equivalence relation and  $\leq$  then induces a partial order (called the Rudin-Keisler order [1, 3, 4]) on the types of ultrafilters in  $\beta\kappa$  (the set of all ultrafilters on  $\kappa$ ).

Throughout this paper, a set of ultrafilters on  $\kappa$  is called *unordered* if its members are pairwise incompatible in the Rudin-Keisler order. Information about this partial order clearly has applications to the study of  $\beta\kappa$  as the Stone-Cech compactification of the discrete space of cardinality  $\kappa$  and to the construction of other counterexamples in topology. An absence of set-theoretic restrictions is especially important.

It has previously been shown [3] that there are  $2^\kappa$  unordered types of ultrafilters on  $\kappa$ . It is the purpose of this paper to present a proof of S. Shelah that there are  $2^{2^\kappa}$  unordered types of ultrafilters on  $\kappa$ .

The *free set lemma* of A. Hajnal [2] says that if  $|X| = \alpha$  and  $\beta < \alpha$  and  $F: X \rightarrow \mathcal{P}(X)$  satisfies  $x \notin F(x)$  and  $|F(x)| < \beta$ , for all  $x \in X$ , then there is a  $Y \subset X$  with  $x \notin F(y)$  and  $y \notin F(x)$  for all  $x$  and  $y$  in  $Y$  and  $|Y| = \alpha$ .

If  $2^{2^\kappa} > (2^\kappa)^+$ , then setting  $X = \beta_\kappa$ ,  $\alpha = 2^{2^\kappa}$ ,  $\beta = (2^\kappa)^+$

and  $F(U) = \{V \in \beta_\kappa \mid V < U\}$  for all  $U \in \beta_\kappa$ , then we have immediately from the free set lemma that there are  $2^{2^\kappa}$  unordered types of ultrafilters on  $\kappa$ . The following theorem thus completes the proof.

*Theorem (Shelah). There are  $(2^\kappa)^+$  unordered types of ultrafilters on  $\kappa$ .*

*Proof.* If  $\mathcal{G} \subset \mathcal{P}(\kappa)$ , let  $\mathcal{G}' = \mathcal{G} \cup \{\kappa - G \mid G \in \mathcal{G}\}$  and  $\mathcal{G}^* = \{\cap K \mid K \subset \mathcal{G}, K \text{ is finite, and } G \in K \text{ implies } (\kappa - G) \notin K \text{ and } G \neq \emptyset\}$ . If  $K = \emptyset$ , then  $\cap K = \kappa$ .

Let  $\mathcal{J}$  be an independent family of subsets of  $\kappa$ : i.e., (1)  $\mathcal{J} \subset \mathcal{P}(\kappa)$ , (2)  $\mathcal{J} = \mathcal{J}'$ , and (3) no term of  $\mathcal{J}^*$  is empty. Choose  $\mathcal{J}$  of cardinality  $2^\kappa$ .

Define  $\Sigma = \{2^\kappa\}$  if  $2^\kappa$  is regular. Otherwise let  $\Sigma$  be a cofinal in  $2^\kappa$  set of uncountable regular cardinals with no limit of members of  $\Sigma$  belonging to  $\Sigma$ .

Our task would be relatively simple if  $2^\kappa$  were regular. Since  $2^\kappa$  may be singular the standard technique of partitioning  $2^\kappa$  into  $\Sigma$  is necessary as is the defining of  $P_\gamma$  below for  $\gamma < 2^\kappa$  and the reindexing of  $\kappa^\kappa$  and  $\mathcal{J}$  in the middle of our inductive construction. For infinite  $\gamma$ , observe by induction that the cardinality of  $P_\gamma$  is  $|\gamma|$ ; only in retrospect is it clear that  $P_\gamma$  is precisely those subsets of  $\kappa$  which might have been used by the  $\gamma$ th stage of our induction.

Index  $\kappa^\kappa = \{g_\gamma \mid \gamma < 2^\kappa\}$  and  $\mathcal{J} = \{F_\gamma \mid \gamma < 2^\kappa\}$ .

We now define  $P_\gamma \subset \mathcal{P}(\kappa)$  for each  $\gamma < 2^\kappa$  by induction.

Let  $Q_\gamma = \cup_{\delta < \gamma} P_\delta$  and  $R_\gamma = \cup_{\delta < \gamma} Q_\delta$ .

If  $T \in R_\gamma^*$ ,  $F \in (Q_\gamma - R_\gamma)$ ,  $f = g_\delta$  for some  $\delta < \gamma$ , and there is an  $S \in (\mathcal{J} - R_\gamma)^*$  such that  $\emptyset \neq (T \cap S) \subset f^{-1}(F)$ ,

then define  $S(T,F,f) = S$  for some such  $S$ . Otherwise  $S(T,F,f)$  is undefined.

Define  $P_\gamma$  to be the set of all  $X \subset \kappa$  such that at least one of the following:

- (1)  $X \in Q'_\gamma \cup Q_\gamma^* \cup \{F_\delta\} \cup \{\kappa - F_\delta\}$  where  $\delta$  is minimal for  $F_\delta \in (J - Q'_\gamma)$ , or
- (2)  $X = f_\delta^{-1}(Y)$  for some  $\delta < \gamma$  and  $Y \in Q_\gamma$ , or
- (3)  $X = S(T,F,f)$  for some  $T \in R_\gamma^*$ ,  $F \in Q_\gamma - R_\gamma$ , and  $f = g_\delta$  for some  $\delta < \gamma$ .

Reindex  $J = \{G_\gamma \mid \gamma < 2^K\}$  in such a way that, if  $\sigma \in \Sigma$ , then  $\{G_\gamma \mid \gamma < \sigma\} = J \cap P_\sigma$ .

*The construction.* By induction for each  $\alpha < (2^K)^+$  we construct an ultrafilter  $U_\alpha$  on  $\kappa$ ; we then prove that the  $U_\alpha$ s are unordered.

So fix  $\alpha < (2^K)^+$  and assume that  $U_\beta$  has been defined for all  $\beta < \alpha$ . Index  $\{\beta < \alpha\} = \{\alpha_\gamma \mid \gamma < 2^K\}$ . Then reindex  $\{\beta < \alpha\} = \{\beta_\gamma \mid \gamma < 2^K\}$ ,  $\kappa^\kappa = \{f_\gamma \mid \gamma < 2^K\}$  and  $\mathcal{P}(\kappa) = \{T_\gamma \mid \gamma < 2^K\}$  in such a way that, if  $\sigma \in \Sigma$ ,  $f = g_\delta$  for some  $\delta < \sigma$ ,  $\beta = \alpha_\rho$  for some  $\rho < \sigma$ , and  $T \in P_\sigma$ , then  $\{\gamma < \sigma \mid \beta_\gamma = \beta, f_\gamma = f, \text{ and } T_\gamma = T\}$  is stationary in  $\sigma$ . Since there are  $\sigma$  disjoint stationary subsets of  $\sigma$ , and  $\{g_\delta \mid \delta < \sigma\}$ ,  $\{\alpha_\rho \mid \rho < \sigma\}$  and  $P_\sigma$  all have cardinality at most  $\sigma$ , this is no problem.

For each  $\gamma < 2^K$  we now inductively construct a filter  $U_\alpha(\gamma)$ ;  $U_\alpha$  will be an extension of  $\cup_{\gamma < 2^K} U_\alpha(\gamma)$  to an ultrafilter.

So assume that  $\gamma < 2^K$  and let  $V_\alpha(\gamma) = \cup_{\delta < \gamma} U_\alpha(\delta)$  be given. Let  $\sigma$  be the minimal member of  $\Sigma$  greater than  $\gamma$ .

Define  $Z_\gamma = \{Z \subset P_\sigma \mid V_\alpha(\gamma) \subset Z, Z - V_\alpha(\gamma) \text{ is finite, } Z$

is a filter, and no term of  $(Z \cup (\mathcal{J} - Z'))^*$  is empty}.

Our induction hypothesis is that  $U_\alpha(\delta) \in Z_\delta$  for all  $\delta < \gamma$ .

Define  $U_\alpha(\gamma) = V_\alpha(\gamma)$  unless for some limit  $\lambda$  we have one of the following cases.

*Case (0).*  $\gamma = \lambda$  and there are  $Z \in Z_\gamma$ ,  $F \in U_{\beta_\lambda}$ , and  $0 \neq Y \in Z^*$  such that  $Y \subset f_\lambda^{-1}(\kappa - F)$ . In this case let  $U_\alpha(\gamma) = Z$  for some such  $Z$ . Observe that  $f_\lambda(U_\alpha) \neq U_{\beta_\lambda}$  in this case.

*Case (1).*  $\gamma = \lambda + 1$ ,  $T_\lambda \in P_\sigma$ ,  $f_\lambda = g_\delta$  for some  $\delta < \sigma$ , and there is an  $F \in ((P_\sigma \cap \mathcal{J}) - V_\alpha(\gamma)')$  such that  $S(T_\lambda, F, f_\lambda)$  is defined. In this case let  $U_\alpha(\gamma) = \{\kappa - F\} \cup V_\alpha(\gamma)$  for some such  $F$ .

*Case (2).*  $\gamma = \lambda + 2$ . Let  $\delta$  be minimal for  $G_\delta \in (\mathcal{J} - V_\alpha(\gamma)')$ ; let  $F$  be the one of  $G_\delta$  and  $(\kappa - G_\delta)$  such that  $f^{-1}(F)$  does not belong to  $U_{\beta_\lambda}$ . Define  $U_\alpha(\gamma) = V_\alpha(\gamma) \cup \{F\}$  in this case. Observe that this case assures us that  $f_\lambda(U_{\beta_\lambda}) \neq U_\alpha$  and that  $U_\alpha(\sigma)' \supset P_\sigma \cap \mathcal{J}$ .

Let  $U_\alpha$  be an arbitrary extension of  $\{U_\alpha(\gamma) \mid \gamma < 2^K\}$  to an ultrafilter. It remains to prove that  $\{U_\alpha \mid \alpha < (2^K)^+\}$  are unordered; (I) and (II) below complete this proof.

Assume  $\beta < \alpha < (2^K)^+$  and  $f \in \kappa^K$ . There are  $\mu$  and  $\eta$  in  $2^K$  and  $\sigma \in \Sigma$  such that  $f = g_\mu$  and  $\beta = \alpha_\eta$ ,  $\mu < \sigma$  and  $\eta < \sigma$ . Let  $\Lambda = \{\lambda < \sigma \mid \lambda \text{ is a limit and } \beta_\lambda = \beta \text{ and } f_\lambda = f \text{ (in the } \alpha \text{ indexing)}\}$ .

(I)  $f(U_\beta) \neq U_\alpha$ .

*Proof.* By our indexing there is a  $\lambda \in \Lambda$  and by case (2)  $f(U_\beta) \neq U_\alpha$ .

(II)  $f(U_\alpha) \neq U_\beta$ .

*Proof.* For  $T \in (U_\alpha \cap P_\sigma)^*$ , let

$$\Delta_T = \{\delta < \sigma \mid S(T, F, f) \text{ is defined for some } F \in ((\mathcal{J} \cap P_\sigma) - P_\delta)\}.$$

*Case (a).* There is a  $T$  with  $\Delta_T = \sigma$ .

Choose  $\lambda \in \Lambda$  with  $T_\lambda = T$ . There is a  $\gamma \in \sigma$  with  $U_\alpha(\lambda) \subset P_\gamma$ .

Choose a limit  $\lambda' < \sigma$  in the  $\beta$  indexing with  $f = f_{\lambda'}$ , and  $T = T_{\lambda'}$ , and  $(\mathcal{J} \cap P_\gamma) \subset V_\beta(\lambda')$ ; by our indexing and case (2) this is possible. Since there is a  $\delta < \sigma$  with  $V_\beta(\lambda') \subset P_\delta$  and  $\Delta_T = \sigma$ , there is an  $F \in ((P_\sigma \cap \mathcal{J}) - V_\beta(\lambda'))$  such that  $S(T, F, f)$  is defined. Thus, by case (1), there is a  $(\kappa - F) \in U_\beta$  for some such  $F$ . Since  $F \notin V_\beta(\lambda') \supset (P_\gamma \cap \mathcal{J})$ ,  $F \in Q_\rho - R_\rho$  for some  $\rho > (\gamma + 1)$ . Thus  $S = S(T, F, f) \in (\mathcal{J} - R_\rho)^* \subset (\mathcal{J} - P_\gamma)^* \subset (\mathcal{J} - U_\alpha(\lambda))^*$ ; also  $S \in P_\sigma$ . Thus by our inductive hypotheses,  $Z = (V_\alpha(\lambda) \cup S) \in Z_\lambda$ . Since  $T \in V_\alpha(\lambda)$ ,  $Y = (T \cap S) \in Z^*$ . Since  $Y \subset f^{-1}(F)$  and  $(\kappa - F) \in U_\beta$ , by case (0), we chose such a  $Z = U_\alpha(\lambda)$ , hence such an  $f^{-1}(F) \in U_\alpha$ . So  $(\kappa - F) \in U_\beta$  implies  $U_\alpha \neq U_\beta$ .

*Case (b).*  $\Delta_T < \sigma$  for all  $T$ .

For each  $\delta < \sigma$  choose  $\delta^* < \sigma$  such that, for all  $T \in U_\alpha(\delta)$ ,  $\Delta_T \subset \delta^*$ ,  $((P_\delta \cap \mathcal{J}) \subset U_\alpha(\delta^*))$  and  $U_\alpha(\delta) \subset P_{\delta^*}$ . Choose  $\lambda \in \Lambda$  such that  $\gamma < \lambda$  implies  $\gamma^* < \lambda$ . Then choose  $F \in (P_\lambda \cap \mathcal{J}) - Q'_\lambda$  and let  $F$  be the one of  $F$  and  $(\kappa - F)$  which belongs to  $U_\beta$ .

If  $(\{f^{-1}(\kappa - F)\} \cup V_\alpha(\lambda)) \in Z_\lambda$ , then, by case (0)  $f(U_\alpha) \neq U_\beta$ .

If  $(\{f^{-1}(\kappa - F)\} \cup V_\alpha(\lambda)) \notin Z_\lambda$ , then there is an  $S \in (\mathcal{J} - V_\alpha(\lambda))^*$  and  $T \in V_\alpha(\lambda)^*$  such that  $\emptyset \neq (S \cap T) \subset f^{-1}(F)$ . Since, for all  $\delta < \lambda$ ,  $(P_\delta \cap \mathcal{J}) \subset U_\alpha(\delta^*)'$  and  $U_\alpha(\delta)' \subset P_{\delta^*}$ ,  $(Q_\lambda \cap \mathcal{J}) \subset V_\alpha(\lambda)'$  and  $V_\alpha(\lambda) \subset Q_\lambda$ . Thus  $F \in (Q_{\lambda+1} - R_{\lambda+1})'$ ,  $S \in (\mathcal{J} - R_{\lambda+1}^*)^*$ , and  $T \in R_{\lambda+1}^*$ . Hence  $S(T, F, f)$  is defined. But  $T \in U_\alpha(\delta)$  for some  $\delta < \lambda$ ,  $\delta^* < \lambda$ , and  $\Delta_T \subset \delta^*$ . Since  $F \notin Q_\lambda$ , this is a contradiction of the definition of  $\Delta_T$ .

### Bibliography

1. C. C. Chang and H. J. Keisler, *Model theory*, North Holland, New York.
2. I. Juhasz, *Cardinal functions in topology*, Math. Centre tracts 34, Amsterdam, 1975 (A. 3.5) 96.
3. K. Kunen, *Ultrafilters and independent sets*, Trans. Amer. Math. Soc. 172 (1972), 299-306.
4. M. E. Rudin, *Partial orders on types in  $\beta\mathbb{N}$* , Trans. Amer. Math. Soc. 155 (1971), 353-362.

The Hebrew University

Jerusalem

and

University of Wisconsin Madison

Madison, Wisconsin 53706