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Suppose that κ is a cardinal. If U and V are ultrafilters on κ and f: $\kappa \to \kappa$ is a function, we say that $f(U) = V$ if $\{f(H) | H \in U\} = V$. We say that $V \leq U$ if there exists an f with $f(U) = V$. We say that U and V are of *the same type* (or $U = V$) if both $V \leq U$ and $U \leq V$. This is an equivalence relation and $\leq$ then induces a partial order (called the Rudin-Keisler order [1,3,4]) on the types of ultrafilters in $\beta\kappa$ (the set of all ultrafilters on κ).

Throughout this paper, a set of ultrafilters on κ is called *unordered* if its members are pairwise incompatible in the Rudin-Keisler order. Information about this partial order clearly has applications to the study of $\beta\kappa$ as the Stone-Cech compactification of the discrete space of cardinality κ and to the construction of other counterexamples in topology. An absence of set-theoretic restrictions is especially important.

It has previously been shown [3] that there are 2^κ unordered types of ultrafilters on κ. It is the purpose of this paper to present a proof of S. Shelah that there are 2^{2^κ} unordered types of ultrafilters on κ.

The *free set lemma* of A. Hajnal [2] says that *if* $|X| = \alpha$ *and* $\beta < \alpha$ *and* F: $X \to \mathcal{P}(X)$ *satisfies* $x \notin F(x)$ *and* $|F(x)| < \beta$, *for all* $x \in X$, *then there is a* $Y \subset X$ *with* $x \notin F(y)$ *and* $y \notin F(x)$ *for all* x *and* y *in* Y *and* $|Y| = \alpha$.

If $2^{2^\kappa} > (2^\kappa)^+$, then setting $X = \beta_\kappa$, $\alpha = 2^{2^\kappa}$, $\beta = (2^\kappa)^+$

and $F(U) = \{V \in \beta_\kappa | V < U\}$ for all $U \in \beta_\kappa$, then we have immediately from the free set lemma that there are 2^{2^κ} unordered types of ultrafilters on κ. The following theorem thus completes the proof.

Theorem (Shelah). *There are* $(2^\kappa)^+$ *unordered types of ultrafilters on* κ.

Proof. If $\mathcal{G} \subset \mathcal{P}(\kappa)$, let $\mathcal{G}' = \mathcal{G} \cup \{\kappa - G | G \in \mathcal{G}\}$ and $\mathcal{G}^* = \{\cap\mathcal{K} | \mathcal{K} \subset \mathcal{G}, \mathcal{K}$ is finite, and $G \in \mathcal{K}$ implies $(\kappa - G) \notin \mathcal{K}$ and $G \neq \emptyset\}$. If $\mathcal{K} = \emptyset$, then $\cap\mathcal{K} = \kappa$.

Let $\mathcal{J}$ be an independent family of subsets of κ: i.e., (1) $\mathcal{J} \subset \mathcal{P}(\kappa)$, (2) $\mathcal{J} = \mathcal{J}'$, and (3) no term of $\mathcal{J}^*$ is empty. Choose $\mathcal{J}$ of cardinaltiy 2^κ.

Define $\Sigma = \{2^\kappa\}$ if 2^κ is regular. Otherwise let Σ be a cofinal in 2^κ set of uncountable regular cardinals with no limit of members of Σ belonging to Σ.

Our task would be relatively simple if 2^κ were regular. Since 2^κ may be singular the standard technique of partitioning 2^κ into Σ is necessary as is the defining of P_γ below for $\gamma < 2^\kappa$ and the reindexing of κ^κ and $\mathcal{J}$ in the middle of our inductive construction. For infinite γ, observe by induction that the cardinality of P_γ is $|\gamma|$; only in retrospect is it clear that P_γ is precisely those subsets of κ which might have been used by the γth stage of our induction.

Index $\kappa^\kappa = \{g_\gamma | \gamma < 2^\kappa\}$ and $\mathcal{J} = \{F_\gamma | \gamma < 2^\kappa\}$.

We now define $P_\gamma \subset \mathcal{P}(\kappa)$ for each $\gamma < 2^\kappa$ by induction. Let $Q_\gamma = \cup_{\delta<\gamma} P_\delta$ and $R_\gamma = \cup_{\delta<\gamma} Q_\gamma$.

If $T \in R_\gamma^*$, $F \in (Q_\gamma - R_\gamma)$, $f = g_\delta$ for some $\delta < \gamma$, and there is an $S \in (\mathcal{J} - R_\gamma')^*$ such that $\emptyset \neq (T \cap S) \subset f^{-1}(F)$, then define $S(T,F,f) = S$ for some such S. Otherwise $S(T,F,f)$ is undefined.

Define P_γ to be the set of all $X \subset \kappa$ such that at least one of the following:

(1) $X \in Q'_\gamma \cup Q^*_\gamma \cup \{F_\delta\} \cup \{\kappa - F_\delta\}$ where δ is minimal for $F_\delta \in (\mathcal{J} - Q'_\gamma)$, or

(2) $X = f_\delta^{-1}(Y)$ for some $\delta < \gamma$ and $Y \in Q_\gamma$, or

(3) $X = S(T,F,f)$ for some $T \in R^*_\gamma$, $F \in Q_\gamma - R_\gamma$, and $f = g_\delta$ for some $\delta < \gamma$.

Reindex $\mathcal{J} = \{G_\gamma | \gamma < 2^\kappa\}$ in such a way that, if $\sigma \in \Sigma$, then $\{G_\gamma | \gamma < \sigma\} = \mathcal{J} \cap P_\sigma$.

The construction. By induction for each $\alpha < (2^\kappa)^+$ we construct an ultrafilter U_α on κ; we then prove that the U_αs are unordered.

So fix $\alpha < (2^\kappa)^+$ and assume that U_β has been defined for all $\beta < \alpha$. Index $\{\beta < \alpha\} = \{\alpha_\gamma | \gamma < 2^\kappa\}$. Then reindex $\{\beta < \alpha\} = \{\beta_\gamma | \gamma < 2^\kappa\}$, $\kappa^\kappa = \{f_\gamma | \gamma < 2^\kappa\}$ and $\mathcal{P}(\kappa) = \{T_\gamma | \gamma < 2^\kappa\}$ in such a way that, if $\sigma \in \Sigma$, $f = g_\delta$ for some $\delta < \sigma$, $\beta = \alpha_\rho$ for some $\rho < \sigma$, and $T \in P_\sigma$, then $\{\gamma < \sigma | \beta_\gamma = \beta$, $f_\gamma = f$, and $T_\gamma = T\}$ is stationary in σ. Since there are σ disjoint stationary subsets of σ, and $\{g_\delta | \delta < \sigma\}$, $\{\alpha_\rho | \rho < \sigma\}$ and P_σ all have cardinality at most σ, this is no problem.

For each $\gamma < 2^\kappa$ we now inductively construct a filter $U_\alpha(\gamma)$; U_α will be an extension of $\cup_{\gamma<2^\kappa} U_\alpha(\gamma)$ to an ultrafilter.

So assume that $\gamma < 2^\kappa$ and let $V_\alpha(\gamma) = \cup_{\delta<\gamma} U_\alpha(\delta)$ be given. Let σ be the minimal member of Σ greater than γ.

Define $\mathcal{Z}_\gamma = \{Z \subset P_\sigma | V_\alpha(\gamma) \subset Z$, $Z - V_\alpha(\gamma)$ is finite, Z

is a filter, and no term of $(Z \cup (\mathcal{J} - Z'))^*$ is empty}.

Our induction hypothesis is that $U_\alpha(\delta) \in \mathcal{Z}_\delta$ for all $\delta < \gamma$.

Define $U_\alpha(\gamma) = V_\alpha(\gamma)$ unless for some limit λ we have one of the following cases.

Case (0). $\gamma = \lambda$ and there are $Z \in \mathcal{Z}_\gamma$, $F \in U_{\beta_\lambda}$, and $0 \neq Y \in Z^*$ such that $Y \subset f_\lambda^{-1}(\kappa - F)$. In this case let $U_\alpha(\gamma) = Z$ for some such Z. Observe that $f_\lambda(U_\alpha) \neq U_{\beta_\lambda}$ in this case.

Case (1). $\gamma = \lambda + 1$, $T_\lambda \in P_\sigma$, $f_\lambda = g_\delta$ for some $\delta < \sigma$, and there is an $F \in ((P_\sigma \cap \mathcal{J}) - V_\alpha(\gamma)')$ such that $S(T_\lambda, F, f_\lambda)$ is defined. In this case let $U_\alpha(\gamma) = \{\kappa - F\} \cup V_\alpha(\gamma)$ for some such F.

Case (2). $\gamma = \lambda + 2$. Let δ be minimal for $G_\delta \in (\mathcal{J} - V_\alpha(\gamma)')$; let F be the one of G_δ and $(\kappa - G_\delta)$ such that $f^{-1}(F)$ does *not* belong to U_{β_λ}. Define $U_\alpha(\gamma) = V_\alpha(\gamma) \cup \{F\}$ in this case. Observe that this case assures us that $f_\lambda(U_{\beta_\lambda}) \neq U_\alpha$ and that $U_\alpha(\sigma)' \supset P_\sigma \cap \mathcal{J}$.

Let U_α be an arbitrary extension of $\{U_\alpha(\gamma) | \gamma < 2^\kappa\}$ to an ultrafilter. It remains to prove that $\{U_\alpha | \alpha < (2^\kappa)^+\}$ are unordered; (I) and (II) below complete this proof.

Assume $\beta < \alpha < (2^\kappa)^+$ and $f \in \kappa^\kappa$. There are μ and η in 2^κ and $\sigma \in \Sigma$ such that $f = g_\mu$ and $\beta = \alpha_\eta$, $\mu < \sigma$ and $\eta < \sigma$. Let $\Lambda = \{\lambda < \sigma | \lambda$ is a limit and $\beta_\lambda = \beta$ and $f_\lambda = f$ (in the α indexing)}.

(I) $f(U_\beta) \neq U_\alpha$.

Proof. By our indexing there is a $\lambda \in \Lambda$ and by case (2) $f(U_\beta) \neq U_\alpha$.

(II) $f(U_\alpha) \neq U_\beta$.

Proof. For $T \in (U_\alpha \cap P_\sigma)^*$, let

$$\Delta_T = \{\delta < \sigma | S(T,F,f) \text{ is defined for some } F \in ((\mathcal{J} \cap P_\sigma) - P_\delta)\}.$$

Case (a). There is a T with $\Delta_T = \sigma$.

Choose $\lambda \in \Lambda$ with $T_\lambda = T$. There is a $\gamma \in \sigma$ with $U_\alpha(\lambda) \subset P_\gamma$.

Choose a limit $\lambda' < \sigma$ in the β indexing with $f = f_{\lambda'}$, and $T = T_{\lambda'}$, and $(\mathcal{J} \cap P_\gamma) \subset V_\beta(\lambda')'$; by our indexing and case (2) this is possible. Since there is a $\delta < \sigma$ with $V_\beta(\lambda')' \subset P_\delta$ and $\Delta_T = \sigma$, there is an $F \in ((P_\sigma \cap \mathcal{J}) - V_\beta(\lambda')')$ such that $S(T,F,f)$ is defined. Thus, by case (1), there is a $(\kappa - F) \in U_\beta$ for some such F. Since $F \not\in V_\beta(\lambda')' \supset (P_\gamma \cap \mathcal{J})$, $F \in Q_\rho - R_\rho$ for some $\rho > (\gamma + 1)$. Thus $S = S(T,F,f) \in (\mathcal{J} - R_\rho)^* \subset (\mathcal{J} - P_\gamma)^* \subset (\mathcal{J} - U_\alpha(\lambda)')^*$; also $S \in P_\sigma$. Thus by our inductive hypotheses, $Z = (V_\alpha(\lambda) \cup S) \in \mathcal{Z}_\lambda$. Since $T \in V_\alpha(\lambda)$, $Y = (T \cap S) \in Z^*$. Since $Y \subset f^{-1}(F)$ and $(\kappa - F) \in U_\beta$, by case (0), we chose such a $Z = U_\alpha(\lambda)$, hence such an $f^{-1}(F) \in U_\alpha$. So $(\kappa - F) \in U_\beta$ implies $U_\alpha \neq U_\beta$.

Case (b). $\Delta_T < \sigma$ for all T.

For each $\delta < \sigma$ choose $\delta^* < \sigma$ such that, for all $T \in U_\alpha(\delta)$, $\Delta_T \subset \delta^*$, $((P_\delta \cap \mathcal{J}) \subset U_\alpha(\delta^*)')$ and $U_\alpha(\delta)' \subset P_{\delta^*}$. Choose $\lambda \in \Lambda$ such that $\gamma < \lambda$ implies $\gamma^* < \lambda$. Then choose $F \in (P_\lambda \cap \mathcal{J}) - Q'_\lambda$ and let F be the one of F and $(\kappa - F)$ which belongs to U_β.

If $(\{f^{-1}(\kappa - F)\} \cup V_\alpha(\lambda)) \in \mathcal{Z}_\lambda$, then, by case (0) $f(U_\alpha) \neq U_\beta$.

If $(\{f^{-1}(\kappa - F)\} \cup V_\alpha(\lambda)) \notin \mathcal{Z}_\lambda$, then there is an $S \in (\mathcal{J} - V_\alpha(\lambda)')^*$ and $T \in V_\alpha(\lambda)^*$ such that $\emptyset \neq (S \cap T) \subset f^{-1}(F)$. Since, for all $\delta < \lambda$, $(P_\delta \cap \mathcal{J}) \subset U_\alpha(\delta^*)'$ and $U_\alpha(\delta)' \subset P_{\delta^*}$, $(Q_\lambda \cap \mathcal{J}) \subset V_\alpha(\lambda)'$ and $V_\alpha(\lambda) \subset Q_\lambda$. Thus $F \in (Q_{\lambda+1} - R_{\lambda+1})$, $S \in (\mathcal{J} - R'_{\lambda+1})^*$, and $T \in R^*_{\lambda+1}$. Hence $S(T,F,f)$ is defined. But $T \in U_\alpha(\delta)$ for some $\delta < \lambda$, $\delta^* < \lambda$, and $\Delta_T \subset \delta^*$. Since $F \notin Q_\lambda$, this is a contradiction of the definition of Δ_T.

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