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LIFTINGS OF SHAPE MORPHISMS TO FINITE COVERS

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1. Introduction

A familiar and useful result from the theory of covering spaces is that corresponding covers of homotopically equivalent "locally nice" connected spaces are homotopically equivalent. In attempting to establish the result of [7], it was first hoped that the shape theoretic version of this result, stated in terms of Fox's theory of overlays [5,6], could be utilized: corresponding overlays of shape equivalent connected metrizable spaces are shape equivalent. This statement is, however, not generally true. An example to show this is indicated by Diagram 5 of [5] (also by Figure 6 of [6]); therein is shown a plane continuum having the (pointed) shape of the wedge of two circles K, whose "universal" overlay is not shape equivalent to that of K.

The implication of the above is that while the Fundamental Lifting Theorem of covering space theory has a nice analogue (Theorem 17° of [6]) in overlay theory for liftings of *maps*, an analogous statement for liftings of *shape morphisms* fails. Working in the pointed shape category, we show in Section 5 of this paper that such a lifting theorem holds for finite overlays (equivalently, finite covers) of compacta. This work is then used in Section 6 to show that corresponding finite connected overlays of shape equivalent pointed continua are pointed shape equivalent. It is natural that our results be stated in terms of *pointed* shape theory because of the strong connection between the notions of overlay and fundamental progroup. We close by remarking on some shape properties, pointed and unpointed, that are passed on by compacta to their finite overlays. Sections 2-4 contain the background material for Section 5.

All spaces considered herein shall be metrizable and all maps between spaces shall be continuous. We shall use I to denote the interval [0,1] and Z to denote the set of positive integers. If $H: X \times I \to Y$ is a homotopy and $t \in I$, we use H_t to denote the level map on X defined by $H_t(x) = H(x,t)$ for all $x \in X$. Although the fundamental group functor is covariant, our liberal use of subscripts makes it notationally convenient to denote the homomorphism induced by the map $f: (X,x) \to (Y,y)$ by $f^{\#}: \pi(X,x) \to \pi(Y,y)$.

2. Some Remarks on Shape Theory

A perusal of the literature will indicate that there now exist many versions of what has become generally known as *shape theory*. Each of these is defined over a specific class of spaces and, for a given probelm to which two or more such theories may be applied, the results given by these may not agree. Most of the extant theories in some sense agree, though, on the class of metric compacta with that defined originally by Borsuk in [1]. This is also the case for *pointed* shape theory (although not, for example, for the shape theory of compact metric pairs). Thus, when attention is restricted to metric compacta, pointed or not, it becomes more or less a matter of individual choice which theory is to be used.

The particular concept of shape that we deal with in this paper is essentially the ANR-systems approach of [8]. As pointed out above, this is purely a matter of choice, and our results could just as well be formulated in any of the other standard theories equivalent to Borsuk's. We leave the details of this conversion to the interested reader.

Let X denote a compactum. For our purposes, an *ANR-system associated with* X will be a sequence $\underline{U}$: $U_1 \supset U_2 \supset \cdots$ of compact ANR's, regarded as an inverse sequence with the inclusions $u_{ij}: U_j \to U_i$ as bonding maps, such that $X = \cap_{i=1}^{\infty} U_i$. It is well known that such systems exist. System maps, etc. are as defined and denoted in [8]. If $U_1, U_2, \cdots$ are connected, then $\underline{U}$ is a *connected* ANR-system. In the pointed case, there is specified a base point $x \in X$ which then becomes the base point of each U_i. All maps and homotopies are then required to respect base points.

3. Algebraic Preliminaries

The classical notions of *covering space* and *fundamental group* are closely interrelated. Of course, the fundamental group has little significance for spaces not locally well behaved. The "corrected" invariant corresponding to the fundamental group is the *fundamental progroup*, described below.

The algebra of pro($\mathcal{G}$), the category of progroups (called *tropes* in [3], [5], and [6]), has been studied in [3]. Since we consider in this paper only shapes of compacta, we need only concern ourselves at this time with the full subcategory of pro($\mathcal{G}$) whose objects are the inverse *sequences*. For convenience then, we disregard the most general setting and provisionally modify some of the notions of [3] to suit our present aims.

Suppose $\underline{G} = \{G_i, g_{ij}\}$ is an inverse sequence of groups and homomorphisms. Then $\underline{H} = \{H_i, h_{ij}\}$ is a *subprogroup* of $\underline{G}$ if H_i is a subgroup of G_i for $i = 1,2,\cdots$ and $h_{ij}(x) = g_{ij}(x)$ for $i \leq j$ and $x \in H_j$. If $\underline{H} = \{H_i, h_{ij}\}$ and $\underline{K} = \{K_i, k_{ij}\}$ are subprogroups of $\underline{G}$, then $\underline{H}$ and $\underline{K}$ are *equivalent*, written $\underline{H} \equiv \underline{K}$, if for each $n \in Z$ there exists $m \in Z$ such that $m \geq n$, $g_{nm}(K_m) \subset H_n$, and $g_{nm}(H_m) \subset K_n$. It is easily verified that $\equiv$ is an equivalence relation and that equivalent subprogroups of $\underline{G}$ are isomorphic objects of pro($\mathcal{G}$).

Now suppose $\underline{G} = \{G_i, g_{ij}\}$ and $\underline{H} = \{H_i, h_{ij}\}$ are inverse sequences of groups and homomorphisms, $\underline{f} = (f, f_i)\colon \underline{G} \to \underline{H}$ is a system map, and $\underline{K} = \{K_i, k_{ij}\}$ is a subprogroup of $\underline{G}$. Let $\underline{f}(\underline{K}) = \{P_i, p_{ij}\}$, where $P_i = f_i(K_{f(i)})$ and $p_{ij}(x) = h_{ij}(x)$ for $i,j \in Z$, $i \leq j$, and $x \in P_j$. Generalizing Proposition 2 on pg. 12 of [3], we have the following.

Proposition 1. *Suppose* $\underline{G}$ *and* $\underline{H}$ *are inverse sequences of groups,* $\underline{f}\colon \underline{G} \to \underline{H}$ *is a system map, and* $\underline{K}$ *is a subprogroup of* $\underline{G}$. *Then* $\underline{f}(\underline{K})$ *is a subprogroup of* $\underline{H}$. *Furthermore, if* $\underline{K}'$ *is a subprogroup of* $\underline{G}, \underline{f}'\colon \underline{G} \to \underline{H}$ *is a system map,* $\underline{K} \equiv \underline{K}'$, *and* $\underline{f} \simeq \underline{f}'$, *then* $\underline{f}(\underline{K}) \equiv \underline{f}'(\underline{K}')$.

Proof. That $\underline{f}(\underline{K})$ is a subprogroup of $\underline{H}$ is immediate from the definition of $\underline{f}(\underline{K})$. To complete the proof, we show (i) $\underline{f}(\underline{K}) \equiv \underline{f}(\underline{K}')$ and (ii) $\underline{f}(\underline{K}') \equiv \underline{f}'(\underline{K}')$. We use notations as in the paragraph preceding the statement of the proposition, and we let $\underline{f}' = (f', f_i')$, $\underline{K}' = \{K_i', k_{ij}'\}$, $\underline{f}(\underline{K}') = \{P_i', p_{ij}'\}$, and $\underline{f}'(\underline{K}') = \{P_i'', p_{ij}''\}$.

To establish (i), let $n \in Z$. Since $\underline{K} \equiv \underline{K}'$, there exists $r \in Z$ such that $r \geq f(n), g_{f(n),r}(K_r) \subset K_{f(n)}'$, and $g_{f(n),r}(K_r') \subset K_{f(n)}$. Now, choose $m \in Z$ so that $f(m) \geq r$. Then $h_{nm}(P_m') = h_{nm}(f_m(K_{f(m)}')) = h_{nm}f_m(K_{f(m)}') = f_n g_{f(n),f(m)}(K_{f(m)}') = f_n g_{f(n),r} g_{r,f(m)}(K'_{f(m)}) \subset f_n g_{f(n),r}(K_r') \subset f_n(K_{f(n)}) = P_n$. Symmetrically, $h_{nm}(P_m) \subset P_n'$, and it follows that $\underline{f}(\underline{K}) \equiv \underline{f}(\underline{K}')$.

To establish (ii), let $n \in Z$. Since $\underline{f} \simeq \underline{f}'$, there exists $r \in Z$ such that $r \geq \max(f(n), f'(n))$ and $f_n g_{f(n),r} = f_n' g_{f'(n),r}$. Now, choose $m \in Z$ so that $m \geq n$ and $r \leq \min(f(m), f'(m))$. Then $h_{nm}(P_m'') = h_{nm}(f_m'(K_{f'(m)}')) = h_{nm} f_m'(K_{f'(m)}') = f_n' g_{f'(n),f'(m)}(K_{f'(m)}') = f_n' g_{f'(n),r} g_{r,f'(m)}(K_{f'(m)}') \subset f_n' g_{f'(n),r}(K_r') = f_n g_{f(n),r}(K_r') \subset f_n(K_{f(n)}') = P_n'$. Symmetrically, $h_{nm}(P_m') \subset P_n''$, and it follows that $\underline{f}(\underline{K}') \equiv \underline{f}'(\underline{K}')$.

Let (X,x) denote a pointed compactum and let $\underline{U} = \{U_i, u_{ij}\}$ be an ANR-system associated with (X,x). The *fundamental pro-group* of (X,x) is the progroup $\underline{\pi}(X,x) = \{\pi(U_i,x), u_{ij}^\#\}$. Evidently this depends on $\underline{U}$ as well as (X,x); we may, however, overlook this since the fundamental progroups of (X,x) associated with any two ANR-systems are isomorphic objects of pro($\mathcal{G}$). If $\underline{U}$ and $\underline{V}$ are ANR-systems associated with (X,x) and (Y,y), respectively, and $\underline{f} = (f, f_i): \underline{U} \to \underline{V}$ is a pointed system map, then $\underline{f}^\# = (f, f_i^\#): \underline{\pi}(X,x) \to \underline{\pi}(Y,y)$ is a system map; it is the system map *induced* by $\underline{f}$, and its homotopy class in pro($\mathcal{G}$) depends only on the homotopy class of $\underline{f}$.

4. Overlays

Most of the classical theorems of covering space theory apply only to spaces which are locally well behaved. In order to expand the domain to which covering space theory may reasonably be applied, Fox [5] introduced the notion of overlay; this concept is somewhat extensively studied in [6], where it is shown that, for metrizable spaces, many of the standard results of covering space theory have rather nice analogues in the theory of overlays.

Let X denote a connected metrizable space. Recall that the map P: $\tilde{X} \to X$ is a *covering* of X if there exist open covers $\mathcal{B} = \{B_\lambda | \lambda \in \Lambda\}$ and $\mathcal{A} = \{A_\lambda^\delta | \lambda \in \Lambda, \delta \in \Delta\}$, of X and $\tilde{X}$, respectively, such that

(i) for all $\lambda \in \Lambda, \mathcal{A}_\lambda = \{A_\lambda^\delta | \delta \in \Delta\}$ is a decomposition of $P^{-1}(B_\lambda)$, and

(ii) for all $\lambda \in \Lambda, \delta \in \Delta, A_\lambda^\delta$ is mapped by P topologically onto B_λ.

We say that P is an *overlaying* of X, and that $\tilde{X}$ is an *overlay* of X, if $\mathcal{B}$ and $\mathcal{A}$ can be so selected that, in addition to (i) and (ii), we also have

(iii) whenever $\lambda, \lambda' \in \Lambda$ and $B_\lambda \cap B_{\lambda'} \neq \emptyset$, each member of $\mathcal{A}_\lambda$ intersects precisely one member of $\mathcal{A}_{\lambda'}$.

The crucial property that distinguishes overlayings from coverings is indicated by the Extension Theorem (Theorem 13 of [6]), which states that if X is a subset of the metrizable space M and P: $\tilde{X} \to X$ is an overlaying, then P extends over some superspace of $\tilde{X}$ to an overlaying of some neighborhood of X in M. Conversely, suppose P: $\tilde{X} \to X$ is a covering which can always be extended over some neighborhood of X in any metrizable superspace of X. We may then embed X as a subset of a locally connected metrizable space M and extend P to a covering $\tilde{P}: \tilde{U} \to U$, where U is an open neighborhood of X in M. Since U is locally connected, it follows from Theorem 3 of [6] that $\tilde{P}: \tilde{U} \to U$ is an overlaying, and it follows easily from this that $P: \tilde{X} \to X$ is an overlaying. Combining these facts, we obtain the following characterization of overlayings.

Theorem 1. *Suppose* X *is a connected metrizable space and* $P: \tilde{X} \to X$ *is a covering. Then* $P: \tilde{X} \to X$ *is an overlaying if and only if whenever* X *is a subset of the metrizable space* M, P *extends to a covering* $\tilde{P}: \tilde{U} \to U$ *of some neighborhood* U *of* X *in* M; *furthermore, the extended covering may be assumed to be an overlaying.*

The Fundamental Theorem of Overlay Theory, Theorem 15^{o} of [6], classifies the d-fold overlayings $P: (\tilde{X},\tilde{x}) \to (X,x)$ of the connected metrizable space X in terms of the representations of the fundamental progroup of (X,x) in the symmetric group Σ_d. According to Theorem 14^{o} of [6], such an overlaying $\tilde{X}$ is associated with a transitive representation in Σ_d if and only if $\tilde{X}$ is *vertically connected*; that is, if and only if it is impossible to express $\tilde{X}$ as the union of disjoint open sets $\tilde{X}_1$ and $\tilde{X}_2$ so that $P|\tilde{X}_i$ is an overlaying of X for i = 1,2. Bozhuyuk, in Chapter V, Corollary 10 of [3], has classified these *transitive* overlayings in terms of certain subprogroups of the fundamental progroup of (X,x).

Our subsequent work shall always involve this case, as is shown by the following result.

Theorem 2. *Suppose* X *is a continuum and* $P\colon \tilde{X} \to X$ *is a finite covering. Then* P *is an overlaying, and* $\tilde{X}$ *is connected if and only if* $\tilde{X}$ *is vertically connected.*

Proof. That P is an overlaying follows from Theorem 3 of [6]. If $\tilde{X}$ is connected, then $\tilde{X}$ must be vertically connected by definition. We shall complete the proof by assuming that $\tilde{X}$ is not connected and establishing that $\tilde{X}$ is not vertically connected.

Suppose then, that $\tilde{X} = \tilde{X}_1 \cup \tilde{X}_2$, where $\tilde{X}_1 \cap \tilde{X}_2 = \emptyset$ and each of $\tilde{X}_1$ and $\tilde{X}_2$ is nonempty and open. We shall show that $P|\tilde{X}_1$ and $P|\tilde{X}_2$ are coverings of X.

Let $i = 1$ or 2, and let $A = \{x \in X |$ there exists an open neighborhood U of x in X evenly covered by $P|\tilde{X}_i\}$. The proof will be complete if we show that $A = X$. For this, it suffices to show that A is nonempty, open, and closed.

To begin, let $y \in \tilde{X}_i$. Since $P\colon \tilde{X} \to X$ is a finite covering, there exists an open neighborhood V of y in $\tilde{X}$ such that (1) $P|V$ is 1-1, (2) $P(V)$ is an open set evenly covered by P, and (3) no one of the homeomorphic copies of V comprising $P^{-1}(P(V))$ intersects both $\tilde{X}_1$ and $\tilde{X}_2$. Then $U = P(V)$ is an open neighborhood of $P(y)$ evenly covered by $P|\tilde{X}_i$. This shows that A is nonempty; moreover, $P(\tilde{X}_i) \subset A$.

That A is open follows from its definition. To see that A is closed, let z be a limit point of A, and let $\{z_n\}_{n=1}^{\infty}$ be a sequence of points of A converging to z. Let y_n be a point of $\tilde{X}_i$ such that $P(y_n) = z_n$, $n = 1,2,\cdots$. By compactness of $\tilde{X}_i$, we may assume that $\{y_n\}_{n=1}^{\infty}$ converges, say to $y \in \tilde{X}_i$. Then $P(y) = z$ and, by the above paragraph, $z \in A$. The proof is complete.

We shall have frequent occasion to use the following fact, the proof of which is elementary and shall be omitted.

Lemma 1. *Suppose* X *is a compactum,* $P\colon \tilde{X} \to X$ *is a finite covering, and* $\tilde{P}\colon \tilde{M} \to M$ *is an extension of* P. *If* U *and* V *are neighborhoods of* $\tilde{X}$ *and* X *in* $\tilde{M}$ *and* M, *respectively, then there exists a neighborhood* W *of* X *in* M *such that* $W \subset V$, $\tilde{P}^{-1}(W) \subset U$, *and* $P'\colon \tilde{P}^{-1}(W) \to W$ *is a covering, where* $P'(z) = \tilde{P}(z)$ *for all* $z \in \tilde{P}^{-1}(W)$.

5. Liftings of Shape Morphisms

Suppose Y is a metrizable space and $Q\colon \tilde{Y} \to Y$ is an overlaying. Then Q induces a shape morphism $\underline{Q}\colon \tilde{Y} \to Y$. If $\underline{f}\colon X \to Y$ is a shape morphism, we say that the shape morphism $\underline{h}\colon X \to \tilde{Y}$ is a *lifting* of $\underline{f}$ (through Q) if $\underline{Qh} \simeq \underline{f}$. The obvious modifications apply to define *lifting* in the case of pointed shape morphisms.

Remark. Throughout this paper a "shape morphism" will be a system map defined on a very special ANR-system. This is, however, only a matter of convenience, and our results may easily be translated into any shape theory equivalent to Borsuk's [1] on metric compacta; cf. second paragraph, Section 2. The above definition would have to be modified to conclude "$\underline{Qh} = \underline{f}$" if, for example, "shape morphism" were to mean "homotopy class of system maps."

We now establish notation to be used throughout the remainder of this section. Through the statement of Theorem 5, Section 6, we deal only with *pointed* shape. We shall, however, suppress base points in many instances in order to simplify the notation. Let (Y,y) be a pointed compactum and $Q\colon (\tilde{Y},\tilde{y}) \to (Y,y)$ a finite covering. Then there exist compact ANR's V_1 and $\tilde{V}_1$ containing Y and $\tilde{Y}$, respectively, such that (1) Y has arbitrarily small closed ANR neighborhoods in V_1 and, (2) there exists a covering $Q_1\colon \tilde{V}_1 \to V_1$ such that $Q_1(z) = Q\,(z)$ for all $z \in \tilde{Y}$. Let $\underline{V}\colon V_1 \supset V_2 \supset \cdots$ and $\underline{\tilde{V}}\colon \tilde{V}_1 \supset \tilde{V}_2 \supset \cdots$ be ANR-systems associated with (Y,y) and $(\tilde{Y},\tilde{y})$, respectively, such that if $i \in Z$, then $\tilde{V}_i = Q_1^{-1}(V_i)$ and $Q_i\colon \tilde{V}_i \to V_i$ is a covering, where $Q_i(z) = Q_1(z)$ for all $z \in \tilde{V}_i$. Let (X,x) be a pointed continuum and let $\underline{U}\colon U_1 \supset U_2 \supset \cdots$ be a connected ANR-system associated with (X,x). All system maps will be defined with respect to the systems $\underline{U}$, $\underline{V}$, and $\underline{\tilde{V}}$. The system map $\underline{Q}$ is simply $(1_Z, Q_i)$.

We are now prepared to state and prove our main results.

Theorem 3 *(Uniqueness). Suppose* (Y,y) *is a pointed compactum,* $Q\colon (\tilde{Y},\tilde{y}) \to (Y,y)$ *is a finite covering,* (X,x) *is a pointed continuum, and* $\underline{f}\colon (X,x) \to (Y,y)$ *is a pointed shape morphism. Then any two liftings of* f *through* Q *are homotopic.*

Proof. Suppose $\underline{h} = (h,h_i)$ and $\underline{g} = (g,g_i)$ are liftings of $\underline{f}$. Then $\underline{Qh} \simeq \underline{f} \simeq \underline{Qg}$. In order to show $\underline{h} \simeq \underline{g}$, fix $i \in Z$. Then there exists $j \in Z$ such that $j \geq \max(h(i),g(i))$ and $Q_i h_i | U_j \simeq Q_i g_i | U_j$ in V_i; that is, there exists a map $H\colon U_j \times I \to V_i$ such that $H_0 = Q_i h_i | U_j$ and $H_1 = Q_i g_i | U_j$. Now, $h_i | U_j$ is a lifting of H_0 through Q_i so, by the Homotopy Lifting Theorem, there exists a map $\tilde{H}\colon U_j \times I \to \tilde{V}_i$ such that

$\tilde{H}_0 = h_i|U_j$ and $Q_i\tilde{H}_1 = H_1 = Q_i g_i|U_j$. By uniqueness of liftings (of maps on pointed continua), $\tilde{H}_1 = g_i|U_j$, and so $h_i|U_j \simeq g_i|U_j$ in $\tilde{V}_i$. This shows $\underline{h} \simeq \underline{g}$.

Theorem 4 (Existence). Suppose (Y,y) *is a pointed compactum,* $Q: (\tilde{Y},\tilde{y}) \to (Y,y)$ *is a finite covering,* (X,x) *is a pointed continuum, and* $\underline{f}: (X,x) \to (Y,y)$ *is a pointed shape morphism. Then* $\underline{f}$ *lifts through* Q *if and only if* $\underline{f}^{\#}(\underline{\pi}(X,x))$ *is equivalent to a subprogroup of* $\underline{Q}^{\#}(\underline{\pi}(\tilde{Y},\tilde{y}))$.

Proof. Suppose first that $\underline{h} = (h,h_i): X \to \tilde{Y}$ is a lifting of $\underline{f} = (f,f_i)$ through Q. Then $\underline{f} \simeq \underline{Qh}$, and so there exist positive integers $j_1 < j_2 < \cdots$ such that for all $i \in Z$, $j_i \geq \max(f(i),h(i))$ and $f_i|U_{j_i} \simeq Q_i h_i|U_{j_i}$ in V_i. If $i \in Z$, let $g_i = Q_i h_i|U_{j_i}: U_{j_i} \to V_i$. Then g_i lifts through Q_i, so $g_i^{\#}(\pi(U_{j_i})) \subset Q_i^{\#}(\pi(\tilde{V}_i))$. Let $H_i = g_i^{\#}(\pi(U_{j_i}))$. Since $f_i u_{f(i),j_i} = f_i|U_{j_i} \simeq g_i, H_i = (f_i u_{f(i),j_i})^{\#}(\pi(U_{j_i}))$. Also, if $i \in Z$, then $f_i|U_{j_{i+1}} \simeq f_{i+1}|U_{j_{i+1}}$ in V_i, and it follows that $v_{i,i+1}^{\#}(H_{i+1}) = v_{i,i+1}^{\#}(f_{i+1}u_{f(i+1),j_{i+1}})^{\#}(\pi(U_{j_{i+1}})) = (v_{i,i+1}f_{i+1}u_{f(i+1),j_{i+1}})^{\#}(\pi(U_{j_{i+1}})) = (f_i u_{f(i),j_{i+1}})^{\#}(\pi(U_{j_{i+1}})) = (f_i u_{f(i),j_i} u_{j_i,j_{i+1}})^{\#}(\pi(U_{j_{i+1}})) \subset (f_i u_{f(i),j_i})^{\#}(\pi(U_{j_i})) = H_i$. The inclusion $v_{i,i+1}^{\#}(H_{i+1}) \subset H_i$ shows that the groups H_i, $i \in Z$, along with the homomorphisms $h_{ij}: H_j \to H_i$ induced by the homomorphisms $v_{ij}^{\#}, i \leq j$, form a subprogroup $\underline{H}$ of $\underline{Q}^{\#}(\underline{\pi}(\tilde{Y}))$. We shall complete the first part of the proof by showing that $\underline{H} \equiv \underline{f}^{\#}(\underline{\pi}(X))$. To verify this, fix $n \in Z$, and let $m = j_n$. Note that (1) $m \geq n$, (2) $v_{nm}^{\#}[f_m^{\#}(\pi(U_{f(m)}))] = (v_{nm}f_m)^{\#}(\pi(U_{f(m)})) = (f_n u_{f(n),f(m)})^{\#}(\pi(U_{f(m)})) = (f_n u_{f(n),m} u_{m,f(m)})^{\#}(\pi(U_{f(m)})) \subset$

$(f_n u_{f(n),m})^{\#}(\pi(U_m)) = H_n$, and (3) $v_{nm}^{\#}[H_m] = v_{nm}^{\#}[(f_m u_{f(m),j_m})^{\#}(\pi(U_{j_m}))] = (v_{nm} f_m u_{f(m),j_m})^{\#}(\pi(U_{j_m})) = (f_n u_{f(n),f(m)} u_{f(m),j_m})^{\#}(\pi(U_{j_m})) \subset f_n^{\#}(\pi(U_{f(n)}))$. Since n was arbitrary, the existence of $m \in Z$ satisfying (1)-(3) shows that $\underline{H} \equiv \underline{f}^{\#}(\underline{\pi}(X))$, and the first portion of the proof is complete.

Now suppose that there exists a subprogroup $\underline{H}$ of $\underline{Q}^{\#}(\underline{\pi}(\tilde{Y}))$ such that $\underline{H} \equiv \underline{f}^{\#}(\underline{\pi}(X))$. Let $\underline{f} = (f, f_i)$. Since $\underline{H} \equiv \underline{f}^{\#}(\underline{\pi}(X))$, there exist positive integers $j_1 < j_2 < \cdots$ such that for all $i \in Z$,

$$(*) \quad v_{i,j_i}^{\#}[f_{j_i}^{\#}(\pi(U_{f(j_i)}))] \subset H_i.$$

If $i \in Z$, let $g(i) = f(j_i)$ and $g_i = v_{i,j_i} f_{j_i}: U_{g(i)} \to V_i$, and let $\underline{g} = (g, g_i)$. It is easily verified that $\underline{g}: \underline{U} \to \underline{V}$ is a system map and that $\underline{g} \simeq \underline{f}$.

If $i \in Z$, then $H_i \subset Q_i^{\#}(\pi(\tilde{V}_i))$. It follows, then, from (*) that g_i may be lifted through Q_i to $h_i: U_{g(i)} \to \tilde{V}_i$. Now, let $h = g$ and let $\underline{h} = (h, h_i)$. We claim that (i) $\underline{h}$ is a system map, and (ii) $\underline{h}$ is a lifting of $\underline{f}$ through Q.

To prove (i), let n and m be positive integers such that $n < m$. Since $\underline{g}$ is a system map, $g_n | U_{g(m)} \simeq g_m$ in V_n. It follows that there exists a homotopy $H: U_{h(m)} \times I \to V_n$ such that $H_0 = g_n | U_{h(m)}$ and $H_1 = v_{nm} g_m$. Now, $h_n | U_{h(m)}: U_{h(m)} \to \tilde{V}_n$ is a lifting of H_0 through Q_n. It follows from the Homotopy Lifting Theorem that there exists a homotopy $\tilde{H}: U_{h(m)} \times I \to \tilde{V}_n$ lifting H through Q_n such that $\tilde{H}_0 = h_n | U_{h(m)}$. Then $\tilde{H}_1$ is a lifting of $v_{nm} g_m$ through Q_n, as is $\tilde{v}_{nm} h_m$. By the uniqueness of liftings of pointed maps, $\tilde{H}_1 = \tilde{v}_{nm} h_m$, and so $h_n | U_{h(m)} \simeq$

$\tilde{v}_{nm}h_m$ in $\tilde{V}_n$. The proof of (i) is complete. To prove (ii), we simply note that $\underline{Qh} = \underline{g} \simeq \underline{f}$.

6. Shapes of Finite Covers

Let (X,x) and (Y,y) be pointed continua, $\underline{f}$: (X,x) → (Y,y) a pointed shape equivalence, and P: $(\tilde{X},\tilde{x}) \to (X,x)$ a finite connected covering. By the results of Chapter V of [3], there exists a finite connected covering Q: $(\tilde{Y},\tilde{y}) \to (Y,y)$ such that $\underline{f}^{\#}\underline{P}^{\#}(\underline{\pi}(\tilde{X})) \equiv \underline{Q}^{\#}(\underline{\pi}(\tilde{Y}))$. We say that P and Q are *corresponding coverings* of (X,x) and (Y,y). If $\underline{g}$: (Y,y) → (X,x) is an inverse of $\underline{f}$, then $\underline{g}^{\#}\underline{Q}^{\#}(\underline{\pi}(\tilde{Y})) \equiv \underline{g}^{\#}\underline{f}^{\#}\underline{P}^{\#}(\underline{\pi}(\tilde{X})) \equiv \underline{P}^{\#}(\underline{\pi}(\tilde{X}))$. By Theorem 4, $\underline{fP}$ and $\underline{gQ}$ can be lifted to $\underline{\tilde{f}}$: $(\tilde{X},\tilde{x}) \to (\tilde{Y},\tilde{y})$ and $\underline{\tilde{g}}$: $(\tilde{Y},\tilde{y}) \to (\tilde{X},\tilde{x})$, respectively. Then $\underline{P\tilde{g}\tilde{f}} \simeq \underline{gQ\tilde{f}} \simeq \underline{gfP} \simeq \underline{P} = \underline{P}\ \underline{id}_{(\tilde{X},\tilde{x})}$. By Theorem 3, $\underline{\tilde{g}\tilde{f}} \simeq \underline{id}_{(\tilde{X},\tilde{x})}$. Similarly, $\underline{\tilde{f}\tilde{g}} \simeq \underline{id}_{(\tilde{Y},\tilde{y})}$. We have, therefore, established the following.

Theorem 5. *Suppose* (X,x) *and* (Y,y) *are shape equivalent pointed continua, and* $(\tilde{X},\tilde{x})$ *and* $(\tilde{Y},\tilde{y})$ *are corresponding finite connected coverings of* (X,x) *and* (Y,y), *respectively. Then* $(\tilde{X},\tilde{x})$ *and* $(\tilde{Y},\tilde{y})$ *are shape equivalent.*

Corollary 1. *Suppose* X *is a compactum having the shape of a finite complex. Then every finite cover of* X *has the shape of a finite complex.*

Proof. Suppose X is a compactum, K is a finite complex such that Sh(X) = Sh(K), and P: $(\tilde{X},\tilde{x}) \to (X,x)$ is a finite covering. We may assume that X and $\tilde{X}$ are connected. Let $k \in K$. Since (K,k) is pointed 1-movable, it follows from the Proposition on pg. 60 of [4] that Sh(X,x) = Sh(K,k). Let Q: $(\tilde{K},\tilde{k}) \to (K,k)$ be a covering of K by the connected complex $\tilde{K}$ such that P and Q correspond. By Theorem 5, $Sh(\tilde{K},\tilde{k}) = Sh(\tilde{X},\tilde{x})$, and consequently $Sh(\tilde{K}) = Sh(\tilde{X})$.

By Corollary 1, the shape property "having the shape of a finite complex" is one which is passed on to finite covers of compacta. Some other shape properties which are passed on in this manner are indicated by the following result; for definitions of these, see [2].

Theorem 6. *Suppose* X *is a compactum,* $P: \tilde{X} \to X$ *is a finite covering,* $x \in X$, $\tilde{x} \in P^{-1}(x)$, *and* $n \in Z$. *Then*

(i) $\tilde{X}$ *is movable if* X *is movable;*

(ii) $(\tilde{X},\tilde{x})$ *is pointed movable if* (X,x) *is pointed movable;*

(iii) $\tilde{X}$ *is an* FANR *if* X *is an* FANR;

(iv) $(\tilde{X},\tilde{x})$ *is a pointed* FANR *if* (X,x) *is a pointed* FANR;

(v) $\tilde{X}$ *is* n-*movable if* X *is* n-*movable;*

(vi) $(\tilde{X},\tilde{x})$ *is pointed* n-*movable if* (X,x) *is pointed* n-*movable.*

Proof. We prove only (i). The proofs of (ii)-(vi) are much the same once we note that to prove (iii) it suffices to show that $\tilde{X}$ is strongly movable (Theorem 7.6, pg. 264 of [2]) and that to prove (iv) it suffices to show that if $\tilde{X} \subset Y \in$ ANR, then the condition $\mathrm{Comph}(Y,\tilde{X})$ is satisfied (cf. the Theorem on pg. 55 of [4]).

To prove (i), assume X is movable and let M and $\tilde{M}$ be ANR-spaces containing X and $\tilde{X}$, respectively, for which there exists a covering $Q: \tilde{M} \to M$ such that $Q(z) = P(z)$ for all $z \in \tilde{X}$. Let U be a neighborhood of $\tilde{X}$ in $\tilde{M}$, and let U' be a neighborhood of X in M such that $Q^{-1}(U') \subset U$ and

$Q': Q^{-1}(U') \to U'$ is a covering, where $Q'(z) = Q(z)$ for all $z \in Q^{-1}(U')$. Since X is movable, there exists a neighborhood V' of X in M such that if W' is any neighborhood of X in M, there exists a homotopy $H: V' \times I \to U'$ such that $H(z,0) = z$ and $H(z,1) \in W'$ for all $z \in V'$. Let $V = Q^{-1}(V')$, and let W be a neighborhood of $\tilde{X}$ in $\tilde{M}$. Let W' be a neighborhood of X in M such that $Q^{-1}(W') \subset W$, and let H be a homotopy as above. Let $G: V \times I \to U'$ be defined by $G(z,t) = H(Q(z),t)$ for all $z \in V$, $t \in I$. Then $G_0(z) = H(Q(z),0) = Q(z)$ for all $z \in V$, and so the inclusion $i_V: V \to Q^{-1}(U')$ is a lifting of G_0 through Q'. By the Homotopy Lifting Theorem, there exists a homotopy $\tilde{G}: V \times I \to Q^{-1}(U')$ such that $\tilde{G}_0 = i_V$ and $Q'\tilde{G} = G$. Let i denote the inclusion $i: Q^{-1}(U') \to U$, and let $F = i\tilde{G}$: $V \times I \to U$. Then $F(z,0) = z$ and $F(z,1) \in Q^{-1}(W') \subset W$ for all $z \in V$. Since U (and later W) was arbitrarily chosen, the existence of V and F shows that $\tilde{X}$ is movable.

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