TOPOLOGY PROCEEDINGS
Volume 3, 1978
Pages 221–226

http://topology.auburn.edu/tp/

# EXPANSIVE MAPPINGS

by

W. R. Utz

**Topology Proceedings**

| | |
|---|---|
| **Web:** | http://topology.auburn.edu/tp/ |
| **Mail:** | Topology Proceedings |
| | Department of Mathematics & Statistics |
| | Auburn University, Alabama 36849, USA |
| **E-mail:** | topolog@auburn.edu |
| **ISSN:** | 0146-4124 |

# EXPANSIVE MAPPINGS

W. R. Utz

## 1. Introduction

In this paper we indicate, in Section 2, how certain primary results for expansive homeomorphisms follow from a theorem proved by modifying the proof of a known theorem. In Section 3 we compare the property of a compact disconnected space carrying an expansive homeomorphism to its being, topologically, a subset of the Cantor discontinuum. Finally, in Section 4 we prove a theorem giving a sufficient condition for a homeomorphism to be expansive on a space if it is expansive on each of two subsets forming a decomposition of the space.

A self-homeomorphism, T, of a metric space X is said to be expansive ([5], wherein the original term "unstable" was used) if there exists a $\delta(X,T) > 0$ such that corresponding to distinct $x,y \in X$, there is an integer $n(x,y)$ for which $\rho(T^n(x),T^n(y)) > \delta$.

## 2.

If X is a metric space and if T is a self-homeomorphism of X, then the orbits of the distinct points $x,y \in X$ are said to be positively (negatively) asymptotic under T if

$$\lim_{\substack{n \to \infty \\ (n \to -\infty)}} \rho(T^n(x),T^n(y)) = 0$$

Let X be an infinite compact metric space, T an expansive self-homeomorphism of X, $A \subset X$ be closed and invariant under T and let X' be the derived set of X.

*Theorem* 1. *If* $A \subset X' \neq \phi$, *then there exists a point of* A *whose orbit is asymptotic in at least one sense to an orbit of a point of* X.

This theorem includes Theorem 2.1 of [5] (let A = X), Lemma 2.3 of [5] (A is a fixed point) and Theorem 2.4 of [5] (A is a periodic orbit). In both [5] and Sol Schwartzman's improvement [2, p. 87] of Theorem 2.1 of [5] it is assumed that X is dense-in-itself. That X only be a non-trivial compact metric space was assumed in Schwartzman's thesis [4]. Schwartzman's theorem gives no advantage, however, in examples such as the following one wherein one may take A = {0}, for instance.

*Example*. Let

$$X = \{0\} \cup \{1\} \cup \{x_i | x_i = \frac{1}{4i}, i = 1,2,3,\cdots\} \cup \{x_i | x_i = (2^i-1)/2^i, i = 2,3,4,\cdots\}$$

where $T: X \to X$ is defined as T(0) = 0, T(1) = 1, T(x) is a shift to the right by one element of X if $x \neq 0,1$ ($\delta = \frac{1}{2} - \varepsilon$ for any $\varepsilon > 0$). One may take A as any one of the three sets {0}, {1}, {0,1}.

*Proof of the Theorem*. The proof may be had by modifying the proof of Theorem 2.1 [5] as follows. $X^2 = X \times X$ becomes $P = X \times A$ and $D = \{z | z = x \times x, x \in X\}$ becomes $D = \{z | z = a \times a, a \in A\}$. The sequence of points $\{z_i\} \subset P - D$ for which $z_i \to z \in D$ may be had as follows. Let $a \in A \cap X'$ and consider a sequence of distinct elements $\{x_i\} \subset X$ for which $x_i \to a$. Then, since each $x_i \neq a$, $x_i \times a \notin D$ and one has $z_i = x_i \times a \to a \times a \in D$ where each $z_i \in P - D$ (it is possible, of course, to have $x_i \in A$).

**3.**

It is well-known [3] that subsets of the symbol space E(C) based on a finite set of symbols, C, is homeomorphic to the Cantor discontinuum. It is also well-known [5] that the shift homeomorphism on E(C) is expansive. In the theorem of this section we show a close relationship between the property of a compact disconnected space carrying an expansive homeomorphism and its being a subset of the Cantor discontinuum.

*Theorem* 2. *If* X *is a compact metric space,* T(X) = X *is an expansive homeomorphism with expansion constant* $\delta$ *and if* X *can be expressed as a finite sum of disjoint closed sets each of diameter less than* $\delta$*, then* X *is topologically contained in the Cantor discontinuum.*

*Proof*. Suppose that X is the union of the disjoint, closed sets $A_1, A_2, \cdots, A_k$. Consider the space, E(C), of all mappings of the integers, I, into the set, C, of symbols $\{1,2,\cdots,k\}$ with metric

$$\tau(f,g) = \frac{1}{1+\max[m|f(i) = g(i) \text{ for } |i| < m]}$$

for $f, g \in E(C)$, $m \in I^+$, $i \in I$. This space is homeomorphic to the Cantor discontinuum [3].

We will now map X homeomorphically into E(C). If $x \in X$, define $\phi: x \to f \in E(C)$ by the requirement that $f(s) = i$, $s \in I$, $i \in C$, if $T^s(x) \in A_i$. To see that $\phi$ is one-to-one, we notice that if $x \neq y$, $x,y \in X$, then, by the expansiveness of T, $\rho(T^n(x),T^n(y)) > \delta$ for some n and so $\phi(x) \neq \phi(y)$.

If $x \in X$ and $\phi(x) = f \in E(C)$, let N, a positive integer, be given. Since X is the union of a finite number of closed sets, $A_i$, each of these sets is also open. For any $j \in \{0,\pm 1,\pm 2,\cdots,\pm N\}$, if $T^j(x) \in A_i$ for some $i \in \{1,2,\cdots,k\}$, it is clear that there exists a neighborhood $V_j$ of x such that $T^j(V_j) \subset A_i$. Now, let $\varepsilon$ be a positive number such that

$$U(x,\varepsilon) = \{y: \rho(x,y) < \varepsilon\} \subset \bigcap\{V_j: -N \leq j \leq N\}$$

If $\rho(x,y) < \varepsilon$ and if $\phi(y) = g$, then $f(j) = g(j)$ for all $j = 0,\pm 1,\pm 2,\cdots,\pm N$ and so $\sigma(f,g) \leq \frac{1}{N+1}$. Thus, $\phi$ is continuous.

Since X is compact, $\phi$ is a homeomorphism. This completes the proof of the theorem.

**4.**

In the next theorem we show that if a homeomorphism is expansive on each of two subsets of a metric space and if one of the subsets consists of a finite number of orbits, then the homeomorphism is expansive on the entire space. An example is given to show the strength of the requirement that the orbits be finite in number in one of the subsets. B. F. Bryant [1] has given a similar theorem.

*Theorem* 3. *Let* $B_1$, $B_2$ *be invariant disjoint subsets of* $B = B_1 \cup B_2$ *under the homeomorphism* $T(B) = B$. *Suppose that* T *is expansive on* $B_1$ *and, also, expansive on* $B_2$. *If* $B_2$ *consists of the orbits of a finite number of points, then* T *is expansive on* B.

*Proof*. Let $\delta_1$ and $\delta_2$, respectively, be expansion constants for T on $B_1$ and $B_2$. Let $\theta = \delta_1/3$. Suppose that $B_2$ consists of orbits of the s points $x_1, x_2, \cdots, x_s$. For each integer i in the interval $1 \leq i \leq s$ there can be at most one $y_i \in B_1$ such that

$$\rho(T^m(x_i), T^m(y_i)) \leq \theta$$

for all $m \in I$. To see this, suppose that $y_i$ and $z_i$ are two different points of $B_1$ with this property. Then,

$$\rho(T^m(y_i), T^m(z_i)) \leq \rho(T^m(y_i), T^m(x_i)) + \rho(T^m(x_i), T^m(z_i)) < \delta_1$$

for all $m \in I$ which contradicts the expansiveness of T on $B_1$.

For each integer i in the interval $1 \leq i \leq s$, select $\alpha_i$ such that $0 < \alpha_i < \rho(x_i, y_i)$ where a companion $y_i$ exists; otherwise select $\alpha_i = \delta_1/3$. Let $\beta$ be any number for which $0 < \beta < \min[\min(\alpha_i), \delta_2]$.

We will now show that $\beta$ is an expansion constant for T on all of B. This is certainly clear for $x,y \in B_1$ or $x,y \in B_2$. Suppose $x \in B_2$ and $y \in B_1$. There exists a $q \in I$ for which $T^q(x) = x_i$ for some i on the interval $1 < i < s$. If $\rho(T^q(x)), T^q(y)) > \beta$, the proof is complete. If $\rho(T^q(x), T^q(y)) = \rho(x_i, T^q(y)) \leq \beta$, then because of the choices of $\alpha_i$ and $\beta$ we have $\beta < \alpha_i$ and so $T^q(y) \neq y_i$. Thus for some N, $\rho(T^N(x_i), T^{N+q}(y)) > \delta_1/3 > \beta$. That is, $T(T^M(x), T^M(y)) > \beta$ for $M = N + q$ to complete the proof of the theorem.

Theorem 3 is not valid if $B_2$ contains an infinite number of orbits. To see this consider the following subset B of $E^3$, let $B_1$ be the family of rays $z = 0$, $y = ix$ for $x > 0$, $i = 1, \frac{1}{2}, \frac{1}{3}, \cdots$. Let $B_2$ be the rays $z = i$, $y = ix$ for $x > 0$,

$i = \frac{1}{2}, \frac{1}{3}, \cdots$. Let T be a shift outward on each ray such that if R denotes the origin (0,0,0) and if $P \in B$, then $\rho(R,T(P)) = 2\rho(R,P)$. Regardless of the $\delta$ selected, there will be rays of $B_1$ and $B_2$ (near the x-axis) for which points are not separated by $\delta$.

**References**


1. B. F. Bryant, *Expansive self-homeomorphisms of a compact metric space*, Amer. Math. Monthly 69 (1962), 386-391.
2. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc., Providence, Rhode Island, 1955.
3. M. Morse and G. A. Hedlund, *Symbolic dynamics*, Amer. J. Math. 60 (1938), 815-866.
4. S. Schwartzman, Dissertation, Yale University, 1952.
5. W. R. Utz, *Unstable homeomorphisms*, Proc. Amer. Math. Soc. 1 (1950), 769-774.


University of Missouri
Columbia, Missouri 65201