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ON THE CHARACTER OF SUPERCOMPACT SPACES

Jan van Mill and Charles F. Mills[1]

1. Introduction, Definitions and Conventions

A collection of subsets $\mathcal{F}$ of a space X is called a *π-network* for $x \in X$ provided that every neighborhood of x contains a member from $\mathcal{F}$. The *supertightness* p(x,X) of x in X is defined to be the least cardinal κ for which every π-network $\mathcal{F}$ for x consisting of finite subsets of X contains a subfamily $\mathcal{F}' \subset \mathcal{F}$ of cardinality $\leq \kappa$ which is a π-network for x. In addition, the *supertightness* p(X) of X is defined by

$$p(X) = \omega \cdot \sup\{p(x,X) \mid x \in X\}.$$

It is clear that $t(X) \leq p(X)$ for every topological space X (for the definitions of cardinal functions such as t,w,d,c,χ see Juhász [7]); in addition the reader can easily verify that $p(X) = t(X,H_f(X))$, where $H_f(X)$ denotes the hyperspace of finite nonempty subsets of X.

For every compact Hausdorff space X and $k \in \omega$ we say that $\text{cmpn}(X) \leq k$ provided that there is an open subbase $\mathcal{U}$ for X such that every covering of X by elements of $\mathcal{U}$ contains a subcovering consisting of at most k elements of $\mathcal{U}$. In addition, $\text{cmpn}(X) = k$ if $\text{cmpn}(X) \leq k$ and $\text{cmpn}(X) \not\leq k$ and $\text{cmpn}(X) = \infty$ in case $\text{cmpn}(X) \not\leq k$ for all $k \in \omega$. Cmpn(X) is called the *compactness number* of X (cf. Bell & van Mill [3]). It is known that for every $k \in \omega$ there is a compact Hausdorff space X_k for which $\text{cmpn}(X_k) = k$; also $\text{cmpn}(\beta\omega) = \infty$ (cf. Bell

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& van Mill [3]). Spaces with compactness number less than or equal to 2 are just the *supercompact spaces* as defined by de Groot in [6]. Many spaces are supercompact, for example all compact metric spaces (cf. Strok & Szymański [14]; elementary proofs of this fact have recently been discovered by van Douwen [4] and Mills [12]). The first examples of non-supercompact compact Hausdorff spaces were found by Bell [1].

In section 2 of the present paper we will prove a theorem from which the following statement is a corollary:

If X is supercompact then $\chi(X) \leq d(X) \cdot p(X)$.

The supercompactness of X is essential; we will give an example of a space X such that $\mathrm{cmpn}(X) = 3$, $d(X) = p(X) = \omega$ and $\chi(X) = 2^{\omega}$. In addition we show that the inequality cannot be sharpened by considering t instead of p. We construct an example of a supercompact space X such that $d(X) = t(X) = \omega$ while $\chi(X) = p(X) = 2^{\omega}$.

We are indebted to Eric van Douwen for some helpful comments.

2. On the Character of Supercompact Hausdorff Spaces

All topological spaces under discussion are assumed to be Tychonoff.

Let X be a set and let κ be a cardinal. We define (as usual)

$$[X]^{\kappa} = \{A \subset X \mid |A| = \kappa\}$$
$$[X]^{<\kappa} = \{A \subset X \mid |A| < \kappa\}$$
$$[X]^{\leq\kappa} = \{A \subset X \mid |A| \leq \kappa\}.$$

Let X be a space, B be a closed subset of X, and Y be the space obtained from X by identifying B to one point. Let

$f: X \to Y$ be the identification. For $\phi \in \{t,p,\chi\}$ let $\phi(B,X) := \phi(f[B],Y)$.

In case X is supercompact, the supercompactness of X can also be described in terms of a closed subbase: a space is supercompact iff it has a closed subbase with the property that any of its *linked* (= every two of its members meet) subcollections has nonvoid intersection. Such a subbase is called *binary*. Without loss of generality we may assume that a binary subbase is closed under arbitrary intersections. Let $\mathcal{S}$ be a binary subbase for X. For $A \subset X$ define $I(A) \subset X$ by

$$I(A) := \cap\{S \in \mathcal{S} \mid A \subset S\}.$$

Notice that $cl_X(A) \subset I(A)$, since each element of $\mathcal{S}$ is closed, that $I(I(A)) = I(A)$ and that $I(A) \subset I(B)$ if $A \subset B \subset X$. The following lemma was proved in van Douwen & van Mill [5]. For the sake of completeness we will give its proof also here.

2.1. *Lemma* (van Douwen & van Mill [5]). *Let $\mathcal{S}$ be a binary subbase for X and let $p \in X$. If U is a neighborhood of p and if A is a subset of X with $p \in cl_X(A)$, then there is a subset B of A with $p \in cl_X(B)$ and $I(B) \subset U$.*

Proof. Since X is regular, p has a neighborhood V such that $p \in cl_X(V) \subset U$. Let $\mathcal{J}$ be the collection of all finite intersections of elements of $\mathcal{S}$. Choose a finite $\mathcal{F} \subset \mathcal{J}$ such that $cl_X(V) \subset \cup\mathcal{F} \subset U$. Now $\mathcal{F}$ is finite, and $A \cap V \subset \cup\mathcal{F}$, and $p \in cl_X(A \cap V)$; hence there is an $S \in \mathcal{F}$ with $p \in cl_X(A \cap V \cap S)$. Let $B := A \cap V \cap S$. Then $p \in cl_X(B)$, and $B \subset A$, and $I(B) \subset S \subset \cup\mathcal{F} \subset U$.

We now can prove the main result of this section.

2.2. *Theorem. Let* Y *be a continuous image of a supercompact space. Then* $\chi(Y) \leq d(Y) \cdot p(Y)$.

Proof. Let $\mathcal{S}$ be a binary subbase for X which is closed under arbitrary intersections and let $f: X \to Y$ be a continuous surjection. Let $\kappa := d(Y) \cdot p(Y)$ and fix a dense subset $D = \{d_\alpha | \alpha < \kappa\}$ of Y. Choose $y \in Y$ and define

$$\mathcal{F} := \{\cup\mathcal{G} | \mathcal{G} \in [\mathcal{S}]^{<\omega} \text{ and } \exists \text{ neighborhood } U \text{ of } y \text{ such that } f^{-1}(U) \subset \cup\mathcal{G}\}.$$

Notice that for every neighborhood U of y there is an $F \in \mathcal{F}$ such that $f^{-1}(y) \subset F \subset f^{-1}(U)$ since $\mathcal{S}$ is a subbase. For each $F \in \mathcal{F}$ let $F := \cup_{i \leq n(F)} S_i^F$, where $S_i^F \in \mathcal{S}$ for all $i \leq n(F)$. For each $\alpha < \kappa$ take $d'_\alpha \in X$ such that $f(d'_\alpha) = d_\alpha$.

Fix $\alpha < \kappa$ and $F = \cup_{i \leq n(F)} S_i^F \in \mathcal{F}$. For each $i \leq n(F)$ pick a point

$$e_i^\alpha \in \cap_{s \in S_i^F} I(\{d'_\alpha, s\}) \cap S_i^F.$$

Notice that, since $\mathcal{S}$ is binary, it is possible to take such a point. Let $E^\alpha(F) := \{e_0^\alpha, \cdots, e_{n(F)}^\alpha\}$. Then $\{f(E^\alpha(F)) | F \in \mathcal{F}\}$ is a collection of finite subsets of Y such that each neighborhood of y contains a member of it. Since $p(y,Y) \leq \kappa$ we can find a subfamily $\mathcal{F}_\alpha \subset \mathcal{F}$ of cardinality at most κ such that each neighborhood of y contains a member of $\{f(E^\alpha(F)) | F \in \mathcal{F}_\alpha\}$.

We claim that

(*) $\cap(\cup_{\alpha<\kappa} \mathcal{F}_\alpha) \cap cl_X\{d'_\alpha | \alpha < \kappa\} = f^{-1}(y) \cap cl_X\{d'_\alpha | \alpha < \kappa\}$

which proves that $\chi(y,Y) \leq \kappa$ since $|\cup_{\alpha<\kappa} \mathcal{F}_\alpha| \leq \kappa \cdot \kappa = \kappa$. To this end, first observe that $f^{-1}(y) \subset \cap(\cup_{\alpha<\kappa} \mathcal{F}_\alpha)$. Assume that (*) is not true; then there is an $x \in (\cap(\cup_{\alpha<\kappa} \mathcal{F}_\alpha) \cap cl_X\{d'_\alpha | \alpha < \kappa\}) - (f^{-1}(y) \cap cl_X\{d'_\alpha | \alpha < \kappa\})$. Then clearly $f(x) \neq y$

and consequently we may take disjoint neighborhoods U and V of, respectively, y and f(x). By lemma 2.1 we can find a subset $D_0' \subset \{d_\alpha' | \alpha < \kappa\}$ such that $x \in I(D_0') \subset f^{-1}(V)$. Pick $d_{\alpha_0}' \in D_0'$ arbitrarily. In addition, take $F \in \mathcal{J}_{\alpha_0}$ such that $E^{\alpha_0}(F) \subset f^{-1}(U)$. Since $x \in \bigcap(\bigcup_{\alpha<\kappa}\mathcal{J}_\alpha)$ we have that $x \in F = \bigcup_{i\leq n(F)} S_i^F$; hence there is an $i_0 \leq n(F)$ such that $x \in S_{i_0}^F$. Then $e_{i_0}^{\alpha_0} \in \bigcap_{s\in S_{i_0}} I(\{d_{\alpha_0}', s\}) \cap S_{i_0}^F \subset I(\{d_{\alpha_0}', x\}) \cap S_{i_0}^F \subset I(D_0') \cap S_{i_0}^F \subset f^{-1}(V)$. This is a contradiction, however, since $e_{i_0}^{\alpha_0} \in f^{-1}(U)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$.

2.3. *Corollary. Let* X *be a supercompact space and let* B *be a closed subset of* X. *Then* $\chi(B) \leq d(X)\cdot p(B,X)$.

We will now describe the examples announced in the introduction. We start with a useful result, the proof of which was suggested to us by Eric van Douwen. Our original proof was much more complicated.

2.4. *Theorem. Let* γX *be a compactification of a separable metric space* X *such that* $\gamma X - X$ *is homeomorphic to the one point compactification of a discrete space. Then* $p(\gamma X) = \omega$.

Proof. Write $\gamma X - X$ as $D \cup \{\infty\}$, where ∞ is the non-isolated point. Evidently $p(x,\gamma X) = \omega$ for all $x \neq \infty$. It remains to show that $p(x,\gamma X) = \omega$. Let $\mathcal{B}$ be a countable base for X closed under finite union.

For $\mathcal{A},\mathcal{C} \subseteq \mathcal{P}(\gamma X)$ and $S \subseteq \gamma X$ we say that $\mathcal{C}$ *covers* $\mathcal{A}$ *(rel* S) if for every neighborhood U of ∞ with $U \supseteq S$ the following holds: if there is there is $A \in \mathcal{A}$ with $A \subseteq U$ then there is

$C \in \mathcal{C}$ with $C \subseteq U$. We say that $\mathcal{C}$ covers $\mathcal{A}$ if $\mathcal{C}$ covers $\mathcal{A}(\text{rel } \emptyset)$.

We prove that $p(\infty, \gamma X) = \omega$ by proving something formally stronger:

(1) for all $\mathcal{J} \subseteq [\gamma X]^{<\omega}$ there is $\mathcal{J}' \in [\mathcal{J}]^{\leq\omega}$ which covers $\mathcal{J}$.

So let $\mathcal{J} \subseteq [\gamma X]^{<\omega}$. For $B \in \mathcal{B}$ and $n \in \omega$ define

$$\mathcal{J}_{B,n} = \{F \in \mathcal{J}: F \cap X \subseteq B, |F \cap D| = n\}.$$

[We do not care if $\infty \in F$ or not.] Using the fact that $\mathcal{B}$ is closed under finite unions, one can easily prove that (1) follows from

(2) for all $B \in \mathcal{B}$ and $n \in \omega$ there is $\mathcal{J}'_{B,n} \in [\mathcal{J}_{B,n}]^{\leq\omega}$ which covers $\mathcal{J}_{B,n}(\text{rel } B)$.

But evidently (2) follows from

(3) for all $n \in \omega$, if $\mathcal{A} \subseteq [D]^n$ then there is $\mathcal{A}' \in [\mathcal{A}]^{\leq\omega}$ which covers $\mathcal{A}$.

We prove (3) with induction on n. For $n = 0$ there is nothing to prove. Suppose (3) holds for a certain $n \in \omega$, and let $\mathcal{A} \subset [D]^{n+1}$. Let $\mathcal{M}$ be a maximal disjoint subfamily. If $\mathcal{M}$ is infinite let $\mathcal{A}'$ be any member of $[\mathcal{M}]^{\omega}$. If $\mathcal{M}$ is finite

$$\mathcal{A}_x = \{A \in \mathcal{A}: x \in A\} \qquad (x \in \cup\mathcal{M})$$ □

For each $x \in \cup\mathcal{M}$ there is $\mathcal{A}'_x \in [\mathcal{A}_x]^{\leq\omega}$ which covers $\mathcal{A}_x$. Now let $\mathcal{A}' = \cup_{x \in \cup\mathcal{M}} \mathcal{A}'_x$.

This theorem gives us our first example.

2.5. *Example*. A compact space X such that $\text{cmpn}(X) = 3$, $d(X) = p(X) = \omega$ while $\chi(X) = 2^{\omega}$.

Indeed, let X be the one point compactification of the Cantor tree ${}^{\omega}2 \cup {}^{\omega}2$ (cf. Rudin [13]). In van Douwen &

van Mill [5] it was shown that this space has compactness number 3 (this was also shown independently by M. G. Bell). Theorem 2.5 gives us $p(X) = \omega$ while clearly $d(X) = \omega$ and $\chi(X) = 2^\omega$.

We will now describe our second example.

2.6. *Example.* A supercompact space Z for which $d(Z) = t(Z) = \omega$ and $\chi(X) = 2^\omega$.

Indeed, let L be the "double arrow line," i.e. the space $[0,1] \times 2$ lexicographically ordered. Let $A \subset L^2$ be the set $\{\langle x,y\rangle | y \geq x\}$. Then set $Z = L^2/A$, and let $\pi\colon L^2 \to X$ be the projection. Since L is first countable, so is L^2; we conclude that $t(L^2) = \omega$. This implies that $t(Z) = \omega$ since π is closed. Clearly $d(Z) = \omega$. Since $L^2 - A$ contains $\{\langle\langle a,1\rangle,\langle a,0\rangle\rangle |$ $a \in [0,1]\}$ as a closed discrete subset of cardinality 2^ω, A is not a G_δ in L^2 so that $\chi(Z) > \omega$. In fact, it is easily seen that $\chi(Z) = 2^\omega$. It remains only to show that X is supercompact.

To this end, let $\mathcal{A}_0$ be the set of all clopen rectangles in L^2 which do not meet A (a rectangle is the product of two intervals). In addition, let $\mathcal{A}_1 := \{[a,b]^2 | [a,b]$ is clopen in L$\}$. It is easily verified that $\{\pi[B] | B \in \mathcal{A}_0 \cup \mathcal{A}_1\}$ is a binary closed subbase for Z.

The above space Z of example 2.7 has another surprising property; it is the continuous image of a normally supercompact space while $\chi(Z) \not\leq d(Z)\cdot t(Z)$. Below we will prove that for every normally supercompact space X the inequality $\chi(X) \leq d(X)\cdot t(X)$ holds. Hence, in contrast with Theorem 2.2, this is not true for continuous images of normally supercompact spaces.

Recall that a *normally supercompact space* is a space X which possesses a binary subbase $\mathcal{S}$ which in addition is *normal*, i.e. for all disjoint $S_0, S_1 \in \mathcal{S}$ there are $T_0, T_1 \in \mathcal{S}$ such that $S_0 \subset T_0 - T_1$, $S_1 \subset T_1 - T_0$ and $T_0 \cup T_1 = X$. This is not such a strange condition, since in van Mill & Schrijver [10] it was shown that if $\mathcal{S}$ is a binary subbase for X then $\mathcal{S}$ is *weakly normal*, i.e. for all disjoint $S_0, S_1 \in \mathcal{S}$ there is a finite covering $\mathcal{M}$ of X by elements of $\mathcal{S}$ such that each element of $\mathcal{M}$ meets at most one of S_0 and S_1. However, the normally supercompact spaces have much stronger properties than the supercompact spaces, see van Mill [9]. We also want to notice that there is a geometric characterization of normally supercompact spaces, see van Mill & Wattel [11].

Since it is easily seen that each product of linearly orderable compact spaces is normally supercompact we see that the space Z of example 2.6 is the continuous image of a normally supercompact space.

2.7. *Lemma. Let* $\mathcal{S}$ *be a binary normal subbase for* X, *let* $x \in X$ *and let* U *be a neighborhood of* x. *Then there is a neighborhood* V *of* x *such that* $x \in V \subset I(V) \subset U$.

Proof. Without loss of generality we may assume that U is open. Let $\mathcal{F} \in [\mathcal{S}]^{<\omega}$ such that $x \notin \cup\mathcal{F} \supset X - U$. For each $F \in \mathcal{F}$ choose $F' \in \mathcal{S}$ such that $x \in \mathrm{int}_X(F')$ and $F' \cap F = \emptyset$. This is possible since $\mathcal{S}$ is normal and since $\{x\} = \cap\{S \in \mathcal{S} \mid x \in S\}$ and since $\mathcal{S}$ is binary. Then $V := \cap_{F\in\mathcal{F}}\mathrm{int}_X(F')$ is as required.

2.8. *Theorem. Let* X *be a normally supercompact space. Then* $\chi(X) \leq d(X) \cdot t(X)$.

Proof. Use Lemma 2.8 and the same technique as in Theorem 2.2.

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