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DISCRETE SEQUENCES OF POINTS

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Abstract

We consider a weak version of R. L. Moore's property D. Roughly speaking, a space X is said to have property D if each discrete (in the locally finite sense) sequence of points in X can be "expanded" to a discrete family of open sets in X. A space is said to have property wD if each discrete sequence has a subsequence which can be "expanded" to a discrete family of open sets. All regular, submetrizable spaces and all realcompact spaces have property wD. In the class of regular spaces in which every point is a G_δ, property wD is both hereditary and ω_1-fold productive. In the class of T_3-spaces, finite to-one perfect maps preserve property D, but do not necessarily preserve property wD (for example, we show that certain finite-to-one perfect images of the Niemytzski plane and of the Pixley-Roy space do not have property wD). Property wD, however, is preserved by n-to-one perfect maps for every positive integer n. Whether every product of perfectly normal T_1-spaces has property wD is independent of the usual axioms of set theory.

1. Introduction

R. L. Moore introduced property D in his book [Mo_2, p. 69] and this concept has been rediscovered and renamed several times since then [GJ, Problem 3L], [Ha], [Miš], [Mo_1]. As far as we know, property wD was first explicitly considered (under different terminology) by K. Morita [Mo_1], and later in a different context by others [vD_1], [V_3]. Both properties deal with countably infinite sets of points which are discrete (in the whole space) in the locally finite sense. Such sets are often called countable, closed discrete sets, and we sometimes refer to them as discrete sequences (for more precise definitions see §2).

Definition 1. A countably infinite discrete set $A \subset X$ *has property* D *in* X provided there exists a discrete family of open sets $\{U_a: a \in A\}$ such that $U_a \cap A = \{a\}$ for all $a \in A$, and the discrete set A is said to have *property* wD *in* X provided there exists an infinite subset of A which has property D in X.

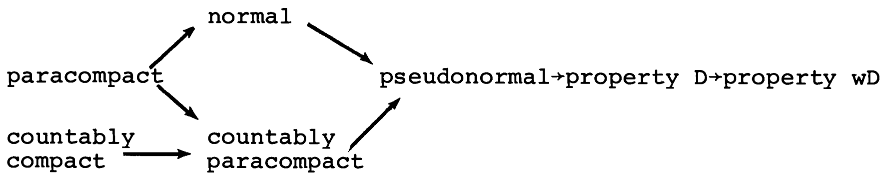
Definition 2. A space X is said to have *property* D (resp. *property* wD) provided that every countably infinite discrete set in X has property D (resp. property wD) in X.

Property D can be considered as a weak form of normality from at least two points of view. Several authors have noticed that a $T_{3\,1/2}$-space X has property D if and only if every discrete sequence in X is C-embedded in X, and that a T_3-space X has property D if and only if every pair of disjoint closed sets, one of which is a discrete sequence, can be separated by disjoint open sets. Analogous characterizations of property wD can be found by starting with either of the two preceding characterizations of property D, and "passing to a subsequence." Most ways of looking at property wD are equivalent in the class of $T_{3\ 1/2}$-spaces, but differ in general. For example the definition of property wD given by R. E. Hodel [Ho] is slightly stronger than the one in this paper because Hodel's definition applies to "sequences having no cluster points" instead of to "discrete sequences" (these two definitions are obviously the same in the class of T_1-spaces).

Since after passing to a subsequence, property wD does the same thing as property D, it retains some of the strength of property D. Indeed, some results which have property D as part of their hypotheses are still true when property D is replaced by property wD. This is the case with Theorems 159 and 161 in Moore's book [Mo_2, p. 70] and Theorems 11 and 14 in [Ha].

In addition to the study of C(X), property wD has been used concerning mapping theorems [SA], cardinal functions in topology [Ho], set theory and topology [vD_1], and countably compact extensions of X in $\beta(X)$ [Mo_1], [Ka_2].

The following diagram of implications (which holds in the class of T_3-spaces) indicates the relation of property wD to several well-known properties.



We believe that property wD is of special interest in spaces in which every point is a G_δ. If we do not have the requirement that every point is a G_δ, then property wD is neither hereditary nor finitely productive in the class of $T_{3\ 1/2}$-spaces. For example, the Tychonoff plank [Wi, p. 122] is easily seen not to have property wD, and yet it is a subspace of a compact T_2-space (clearly, every (countably) compact space has property wD). Thus, property wD is not hereditary in the class of $T_{3\ 1/2}$-spaces. We can also use known examples to show that property wD is not productive in that class either. It is known [GS, Example 5.2] that there exist two countably compact spaces X and Y (both subspaces of $\beta(\omega)$) with the property that $X \times Y$ is pseudocompact (i.e., there exists no infinite locally finite family of open subsets of $X \times Y$) but $X \times Y$ is not countably compact (so $X \times Y$ contains a discrete sequence). Thus, both X and Y have property wD but $X \times Y$ does not.

Our principal theorems are concerned with the preservation of property wD under perfect maps, and with finding classes of spaces in which property wD is hereditary or is productive to some extent. We originally proved several of our theorems in the class of T_3-spaces in which every point is a G_δ, but discovered that we did not require the full strength of that class. For these results, we only need the following concept (which is known under several names): A point x in a space X is called a *Hausdorff*-G_δ provided that $\{x\}$ is the intersection of countably many closed neighborhoods of x (we assume the open neighborhoods are nested).

In §3 we prove

Theorem A. *In the class of spaces in which every point is a Hausdorff*-G_δ, *property* wD *is* ω_1-*fold productive.*

In contrast to this we note that even in the class of first countable, $T_{3\ 1/2}$-spaces, property D is not productive. For example, the Sorgenfrey line S [Wi, 4.6] is a first countable, $T_{3\ 1/2}$-space which has property D (S is a Lindelöf space) but $S \times S$ does not have property D (Sorgenfrey's proof [So] that $S \times S$ is not normal uses two disjoint closed discrete sets, one of which is countable).

The space $S \times S$ has property wD (by Theorem A, or by Corollary 3.5 which states that every submetrizable T_3-space has property wD); so $S \times S$ is an example of a space which has property wD but does not have property D. For an easy example of a Moore space which does not even have property wD we may take the well-known example $\mathcal{N} \cup \mathcal{R}$ of S. Mrówka [Mr_1] (this space is called Ψ in [GJ]). Recall that this space consists of a countable discrete space $\mathcal{N}$ and a maximal family $\mathcal{R}$ of almost disjoint subsets of $\mathcal{N}$. The points of $\mathcal{N}$ are isolated and a local base for a point R in $\mathcal{R}$ consists of all sets of the form $\{R\} \cup T$ where T is a cofinite subset of R. The space $\mathcal{N} \cup \mathcal{R}$ is pseudocompact but not countably compact; so it does not have property wD.

In §4. we prove

Theorem B. *In the class of Urysohn spaces in which every point is a Hausdorff*-G_δ, *property* wD *is hereditary.*

Again note the contrast in behavior of property wD with that of property D: Property D is not hereditary in the class of first countable, $T_{3\ 1/2}$-spaces. To see this, let I^* denote the top and bottom lines of the lexicographically ordered square (cf. [V_1]). We may consider the Sorgenfrey line S as a subspace of I^* and therefore $S \times S$ as a subspace of $I^* \times I^*$. Since I^* is a first countable, compact T_2-space, so is $I^* \times I^*$. Thus $I^* \times I^*$ has property D, but its subspace $S \times S$ does not.

In §5, we give several results concerning perfect maps. We mention one theorem and one example here.

Theorem C. *Let* $f: X \to Y$ *be a perfect map from a space* X *which has property* wD *onto a* T_3-*space* Y. *If there exists a positive integer* n *such that for all* $y \in Y$ *we have* $|f^{-1}(y)| \leq n$, *then* Y *has property* wD.

The restriction in Theorem C on the cardinality of point inverses is needed. Example 5.6 shows that there exists a finite-to-one perfect map defined on the Niemytzki plane (which is a submetrizable $T_{3\ 1/2}$-space) whose image does not have property wD. (This image space is an example of a separable Moore space which does not have property wD).

In §6, we show that Example 5.6 and similar examples (e.g., the Pixley-Roy space [PR]) can be used with property wD in the study of perfect images of submetrizable and real-compact spaces.

Some of the results in this paper were announced in [V_3].

2. Preliminaries

We first recall several definitions.

2.1 *Definitions.* A family $\mathcal{F}$ of subsets of a topological space X is said to be *locally finite at a point* x *in* X provided there exists an open set U containing x which intersects at most a finite number of elements of $\mathcal{F}$. If there exists an open neighborhood of x which intersects at most one member of $\mathcal{F}$, we say that $\mathcal{F}$ is *discrete at* x. A subset $A \subset X$ is called a *discrete set* if the family $\{\{a\}: a \in A\}$ is discrete at every point of X. In this paper we are concerned with the case where A is countable, and we sometimes say that A is a *discrete sequence.* In other words, a discrete sequence of points is a one-to-one indexing of a countably infinite discrete set.

We note that a space in which every point is a G_δ, is a T_1-space, and that in a T_1-space every subset of a discrete set of points is a closed set.

We make the following definition in order that we may consider in detail a standard type of construction.

2.2 *Definition.* A point p in a space X is called a *Hausdorff point* (resp. *Urysohn point*) provided that for every $x \in X \setminus \{p\}$ there exist open sets U, V such that $x \in U$, $p \in V$, and $U \cap V = \emptyset$ (resp. $\overline{U} \cap \overline{V} = \emptyset$).

Clearly, a point which is a Hausdorff-G_δ is a Hausdorff point (this is why we use the term "Hausdorff-G_δ"). Recall that a space is said to be a *Urysohn space* if every pair of distinct points have disjoint closed neighborhoods.

2.3. *Lemma. Let* A *be a countably infinite subset of a space* X, *and let* $p \in \overline{A}\setminus A$. *If* p *is both a Hausdorff*-G_δ *and a Urysohn point in* X, *then there exists an infinite sequence* $\{a_i: i < \omega\}$ *in* A *and a family* $\mathcal{V} = \{V_i: i < \omega\}$ *of open sets in* X *such that* $a_j \in V_i$ iff $i = j$, *and* $\mathcal{V}$ *is discrete at each point of* $X\setminus\{p\}$.

Proof. Let $\{p\} = \bigcap\{\overline{G}_i: i < \omega\}$ where G_i is an open set containing p for each $i < \omega$. Pick $a_0 \in G_0 \cap A$, and let V_0 and U_0 be open sets with $a_0 \in V_0$, $p \in U_0$, and $\overline{V}_0 \cap \overline{U}_0 = \emptyset$. There exists a positive integer n_0 such that $a_0 \notin G_{n_0}$. Pick $a_1 \in (U_0 \cap G_{n_0}) \cap A$ (so $a_0 \neq a_1$) and let V_1 and U_1 be open sets such that $a_1 \in V_1 \subset U_0 \cap G_{n_0}$, $p \in U_1 \subset U_0$, and $\overline{V}_1 \cap \overline{U}_1 = \emptyset$. Thus $\overline{V}_0 \cap \overline{V}_1 = \emptyset$. Continue in this way to construct the sequence $\{a_1: i < \omega\}$ and the family of open sets $\mathcal{V} = \{V_i: i < \omega\}$ whose closures are mutually disjoint and such that $a_i \in V_i \subset G_{n_{i-1}}$ (where $n_0 < n_1 < \cdots < n_{i-1} < n_i < \cdots$). The condition $V_i \subset G_{n_{i-1}}$ implies that $\mathcal{V}$ is locally finite, hence discrete, in $X\setminus\{p\}$.

2.4. *Remark.* In Lemma 2.3, if we assume that p is a Hausdorff-G_δ but not necessarily a Urysohn point, then we can proceed in a manner similar to the proof of Lemma 2.3, and get the family $\mathcal{V} = \{V_i: i < \omega\}$ to be a locally finite family of mutually disjoint open sets, but since the closures of the members of $\mathcal{V}$ are not necessarily disjoint, it does not follow that $\mathcal{V}$ is a discrete family. This situation occurs in Bing's countable, connected Hausdorff space [B].

Likewise, if we only assume in Lemma 2.3 that the point p is a G_δ and a Urysohn point, then we can get the elements of $\mathcal{V}$ to have mutually disjoint closures, but we cannot necessarily get $\mathcal{V}$ to be locally finite at every point in $X\backslash\{p\}$. This situation occurs in a space considered by R. M. Stephenson, Jr. [St., Example 5] which is a Urysohn space in which every point is a G_δ but one point is not a Hausdorff-G_δ.

2.5. *Lemma. A regular* T_1*-space* X *has property* wD *if and only if* X *satisfies the following condition: For every countably infinite discrete set* $A \subset X$*, there exists an infinite locally finite family* $\{V_i: i < \omega\}$ *of open sets in* X *such that* $V_i \cap A \neq \emptyset$ *for infinitely many* $i < \omega$.

Proof. If X has property wD, it satisfies the condition without any assumption concerning separation axioms. We assume the condition holds and that X is a regular T_1-space, and show that X has property wD. By passing to a subsequence of A, and by using the fact that every subset of A is closed in the T_1-space X, we may assume that $V_i \cap A$ consists of a single point, call it a_i, for all $i < \omega$. Let $n_0 = 0$. There exists a positive integer $n_1 < \omega$ and an open neighborhood U_{n_0} of a_0 such that $U_{n_0} \cap V_i = \emptyset$ if $i \geq n_1$. Proceeding by induction, we get a locally finite family $\{U_{n_i} \cap V_{n_i}: i < \omega\}$ of mutually disjoint open sets such that $a_{n_i} \in U_{n_i} \cap V_{n_i}$ for all $i < \omega$. Since X is regular there exist open sets W_i such that $a_{n_i} \in W_i \subset \overline{W}_i \subset U_{n_i} \cap V_{n_i}$. Then $\{W_i: i < \omega\}$ is a discrete family of open sets in X such that $W_i \cap A = \{a_{n_i}\}$ for all $i < \omega$. Thus X has property wD. This completes the proof.

Recall that a space is called *feebly compact* provided that every locally finite family of open sets in the space is finite, and that feeble compactness is equivalent to pseudocompactness in $T_{3\ 1/2}$-spaces.

2.6. *Remark.* A T_1-space is countably compact if and only if it is feebly compact and has property wD.

This can be considered as a slight generalization of [GJ, 3L, 5].

A space X with topology T is called *submetrizable* provided there exists a metrizable topology M on X such that $M \subset T$. We recall a well-known example of a non-regular, submetrizable space. Let M denote the usual topology on the closed unit interval [0,1] and let T be the topology on [0,1] having as a subbase

$$M \cup \{[0,1] \setminus \{1/n: 1 \leq n < \omega\}\}$$

The topology T is obviously submetrizable and not regular. This is an example of a submetrizable space which does not have property wD. One way to see this is to note that this space is feebly compact but not countably compact.

3. Some Basic Results

3.1. *Proposition. If* X *is a first countable, Hausdorff space having property* wD, *then* X *is regular.*

Proof. If X is not regular, then there exist a closed set F and a point $p \notin F$ such that for every neighborhood U of p, $\overline{U} \cap F \neq \emptyset$. Let $\{U_n: n < \omega\}$ be a nested local base at p such that $\{p\} = \cap\{\overline{U}_n: n < \omega\}$. Clearly, for infinitely many $n < \omega$, we have $(\overline{U}_n \setminus \overline{U}_{n+1}) \cap F \neq \emptyset$, and by the nesting, these subsets of F are mutually disjoint. By passing to a subsequence, we may assume that $(\overline{U}_n \setminus \overline{U}_{n+1}) \cap F \neq \emptyset$ for all $n < \omega$. Pick a sequence $a_n \in (\overline{U}_n \setminus \overline{U}_{n+1}) \cap F$ of distinct points. Now $A = \{a_n: n < \omega\}$ is a discrete sequence in X, and hence by property wD (and by passing to a subsequence) we may assume there exists a discrete family $\mathcal{V} = \{V_n: n < \omega\}$ of open sets in X such that $V_n \cap A = \{a_n\}$ for all $n < \omega$. Since $\{U_n: n < \omega\}$ is still a local base for p, there exists $k < \omega$ such that U_k intersects at most one member of $\mathcal{V}$. On the other hand, for all $n < \omega$, $\overline{U}_n \cap V_n \neq \emptyset$; so $U_n \cap V_n \neq \emptyset$. Thus, for $n \geq k$, $U_k \cap V_n \neq \emptyset$. This is a contradiction.

The hypothesis "Hausdorff" cannot be weakened to "T_1" in Proposition 3.1, as the example of the cofinite topology on ω shows.

Proposition 3.1 can be considered as a generalization of C. Aull's result [A_1] that a first countable, countably paracompact T_2-space is regular. The following example, which is an elaboration of Aull's example [A_2, Example 1], shows that "first countable" cannot be in 3.1.

3.2. *Example.* A submetrizable space which has property wD and is not regular.

The set X consists of $\omega_1 \times (\omega + 1)$ together with a point p not in $\omega_1 \times (\omega + 1)$. All points of $\omega_1 \times \omega$ are to be isolated, and a point (α,ω) where $\alpha < \omega_1$ has basic neighborhoods of the form

$$V_n(\alpha,\omega) = \{(\alpha,m): n \leq m \leq \omega\}$$

Let $\{P_n: n < \omega\}$ be a partition of ω_1 into countably many uncountable sets. Define the basic neighborhoods of p as follows: For every $\alpha < \omega_1$ and $n < \omega$

$$W(\alpha,n) = \{(\beta,i): \alpha \leq \beta < \omega_1,\ n \leq i < \omega,\ \beta \in \bigcup_{j\geq n} P_j\} \cup \{p\}.$$

Since each $\overline{W(\alpha,n)}$ contains points whose second coordinate is ω, the space X is not regular. To see that X is submetrizable, note that the sets $\overline{W(\alpha,n)}$ are clopen in X, and that $\{\overline{W(0,n)}: n < \omega\}$ serves as a base for p in a metrizable topology on X in which all the other points have their same neighborhoods. That the space X has property wD (even property D) follows from the fact that for every countable set $A \subset X$ with $p \notin \overline{A}$, there exists $a < \omega_1$ such that $A \cap \overline{W(\alpha,0)} = \emptyset$.

It is obvious that property wD is closed-hereditary, i.e., inherited by closed subspaces. We now consider the hereditary nature of property wD.

3.3. *Proposition. Property* wD *is hereditary in the class of Urysohn spaces in which every point is a Hausdorff-* G_δ.

Proof. Let $Y \subset X$, and let A be a countably infinite discrete set in the subspace Y. If A is also discrete in X, there is nothing to prove; so we assume that there is a point $p \in X\backslash Y$ such that $p \in \overline{A}\backslash A$. By Lemma 2.3, there is a sequence of distinct points $\{a_i: i < \omega\} \subset A$ and a family $\mathcal{V} = \{V_i: i < \omega\}$ of open sets with $V_i \cap A = \{a_i\}$ for all $i < \omega$, and such that $\mathcal{V}$ is discrete at each point of $X\backslash\{p\}$. Since $p \notin Y$, the family $\{V_i \cap Y: i < \omega\}$ is discrete in Y, and this completes the proof.

It follows from Proposition 3.3 that property wD is hereditary in the class of regular spaces in which every point is a G_δ. By Proposition 3.1, therefore, property wD is hereditary in the class of first countable, Hausdorff spaces. We have an example which shows that property wD is not hereditary in the class of Urysohn spaces in which every point is a G_δ. This leaves us with the following

Problem. Is property wD hereditary in the class of spaces in which every point is a Hausdorff-G_δ?

The example ([0,1], T) given in §2 is a non-regular, submetrizable space, which does not have property wD.

The next result shows, however, that every regular, submetrizable space has property wD.

3.4. *Proposition. Let* (X,S) *be a space having property* wD *in which every point is a Hausdorff-*G_δ. *If* T *is a regular topology on* X *which is finer than* S (i.e., $T \supset S$) *then* (X,T) *has property* wD.

Proof. Let $A \subset X$ be a countably infinite discrete set in the space (X,T). If A is also discrete in the space (X,S) there is nothing to prove; so we assume that A is not a discrete set in (X,S). By passing to a subsequence (if necessary), we may assume that there exists a point $p \in Cl_S(A) \setminus A$. By Remark 2.4, there exists an infinite set $\{a_i: i < \omega\} \subset A$ and a family $\{W_i: i < \omega\} \subset S$ which is locally finite in (X,S) at every point of $X\setminus\{p\}$, and such that $a_j \in W_i$ iff $j = i$. Since (X,T) is regular, there exists $U \in T$ such that $p \in U$ and $Cl_T(U) \cap A = \emptyset$. Put $V_i = (W_i \setminus Cl_T(U))$ for all $i < \omega$. Then $\{V_i: i < \omega\}$ is a locally finite family of open sets in (X,T) such that $V_i \cap A \neq \emptyset$ for all $i < \omega$. Lemma 2.5 implies that (X,T) has property wD, and this completes the proof.

3.5. *Corollary. Every regular, submetrizable space has property* wD.

3.6. *Corollary. Let* X *be a first countable, submetrizable space. Then* X *has property* wD *if and only if* X *is regular.*

Example 3.2 shows that "first countable" cannot be deleted from Corollary 3.6.

4. Products

The theory of products of spaces having property wD is closely related to the theory of products of countably compact and feebly compact spaces.

4.1. *Lemma. Let* $\{X_\alpha: \alpha < \kappa\}$ *be a family of* T_1*-spaces having property* wD, *let* A *be an infinite countable discrete subset of* $X = \Pi\{X_\alpha: \alpha < \kappa\}$ *and put* $Y_\alpha = Cl_{X_\alpha}(\pi_\alpha(A))$, *where* π_α *is the usual projection map* $\pi_\alpha: X \to X_\alpha$.

(a) *If there exists* $\alpha < \kappa$ *such that* Y_α *is not feebly compact, then* A *has property* wD *in* X.

(b) *Conversely, if* A *has property* wD *in* X, *and each point in each* X_α *is a Hausdorff-*G_δ, *then there exists* $\alpha < \kappa$ *such that* Y_α *is not feebly compact.*

Proof of (a). If Y_α is not feebly compact, there exists an infinite locally finite family $\{U_i: i < \omega\}$ of open sets in Y_α. Since $\pi_\alpha(A)$ is dense in Y_α, we may pick by induction an infinite sequence $x_{i_j} \in U_{i_j} \cap \pi_\alpha(A)$. Since X_α is T_1, the locally finite set $\{x_{i_j} : j < \omega\}$ of points is a discrete sequence in Y_α and in X_α. By applying property wD and passing to a subsequence, we may assume that there exists a discrete family $\{V_i : i < \omega\}$ of open sets in X_α such that $x_{i_j} \in V_j$ for all $j < \omega$. It is easy to check that $\{\pi^{-1}(V_j) : j < \omega\}$ is discrete in X and that each $\pi_\alpha^{-1}(V_j) \cap A \neq \emptyset$. Since every subset of A is closed in X, we can find an open subset of each $\pi^{-1}(V_j)$ that contains exactly one point of A.

Proof of (b). Suppose A has property wD in X. Then no subspace of X can both contain A and be feebly compact. If each Y_α is feebly compact, then by Remark 2.6 each Y_α is countably compact (since property wD is hereditary to closed subspaces). Further, since each point in Y_α is a Hausdorff-G_δ, Y_α is first countable; hence sequentially compact. By Theorem 5.2 in [SS] every product of sequentially compact spaces is feebly compact. Thus $\Pi\{Y_\alpha : \alpha < k\}$ is feebly compact and contains A. This is a contradiction.

4.2. *Theorem. If* $\{X_\alpha : \alpha < \omega_1\}$ *is a family of spaces having property* wD, *and in which every point is a Hausdorff-*G_δ, *then* $\Pi\{X_\alpha : \alpha < \omega_1\}$ *has property* wD.

Proof. Let A be an infinite countable discrete set in $X = \Pi\{X_\alpha : \alpha < \omega_1\}$. By Lemma 4.1 (a), it suffices to show that there exists some Y_α which is not feebly compact. If this is false, then every Y_α is feebly compact, and proceeding as before, we see that every Y_α is sequentially compact. By a theorem of Scarborough and Stone [SS. Theorem 5.5], every ω_1-fold product of sequentially compact spaces is countably compact; so A is contained in a countably compact subspace of X. This contradicts the assumption that A is discrete in X.

The next result does not require that points be G_δ-sets.

4.3. *Theorem. Let* P *be a topological property which is hereditary to closed subspaces, and such that every countably compact space having property* P *is compact. Then every product of* T_1*-spaces, all of which have both property* P *and property* wD*, has property* wD.

Proof. The proof is similar to that of Theorem 4.2, but uses the Tychonoff product theorem [W_i, 17.8] rather than the theorem of Scarborough and Stone.

We state a few consequences of this result.

4.4. *Corollary.*

(a) *Every product of metric spaces has property* wD.

(b) *Every realcompact space has property* wD.

(c) *Every product of regular,* σ*-spaces having property* wD*, has property* wD.

(d) (MA + ¬CH). *Every product of perfect,* T_3*-spaces having property* wD *has property* wD.

(e) (MA + ¬CH). *Every product of perfectly normal* T_2*-spaces has* wD.

Proof. For (d) use the theorem of W. Weiss [W] which states: (MA + ¬CH) implies that every perfect, countably compact T_3-space is compact. Then (e) is a consequence of (d).

Thus in the class of spaces in which every point is a Hausdorff-G_δ, there are many families whose products have property wD. Finding a family in that class whose product does not have property wD is equivalent to solving a well-known open problem in general topology.

4.5. *Lemma. The following are equivalent.*

(1) *There exists a family* $\{X_\alpha: \alpha < \kappa\}$ *of spaces having* wD, *and in which every point is a Hausdorff-*G_δ, *such that* $X = \Pi\{X_\alpha: \alpha < \kappa\}$ *does not have property* wD.

(2) *There exists a family* $\{Y_\alpha: \alpha < \kappa\}$ *of first countable, sequentially compact* T_2*-spaces whose product* $Y = \Pi\{Y_\alpha: \alpha < \kappa\}$ *is not countably compact.*

Proof. (1) → (2). If X does not have property wD, there exists an infinite closed discrete set $A \subset X$ such that A does not have property wD in X. By Lemma 4.1(a) each $Y_\alpha = Cl_{X_\alpha}(\pi_\alpha(A))$ is feebly compact and therefore sequentially compact and first countable. Since $A \subset Y = \Pi\{Y_\alpha: \alpha < k\}$ we see that Y is not countably compact.

(2) → (1). By Theorem 5.2 of [SS] the product space Y is feebly compact but by hypothesis not countably compact. Thus Y does not have wD by Remark 2.6.

Using this lemma, we can point to product spaces which do not have property wD. Under certain set theoretic assumptions such as CH, several authors have given examples to show that the product of first countable, sequentially compact, T_3-spaces need not be countably compact. As we noted in 4.5, such products do not have property wD because they are feebly compact. Using the set theoretic assumption $\diamondsuit$ and the technique of A. Ostaszewski [O], we constructed [V_2] a family of first countable, perfectly normal, sequentially compact

T_2-spaces whose product is not countably compact. By combining this example with Corollary 4.4(d) we have

4.6. *Proposition. The statement "every product of perfectly normal,* T_2*-spaces has property* wD*" is independent of the usual axioms of set theory* (ZFC).

The theory of box products of spaces having property wD has been considered by Eric van Douwen, who also proved independently some of the results in this section [vD_3].

5. Perfect maps

In this section we show that in the class of T_3-spaces, property wD is not necessarily preserved by finite-to-one perfect maps unless there is an upper bound on the cardinality of the point inverses. On the other hand, property D is preserved by all finite-to-one perfect maps.

5.1. *Proposition. Let* $f: X \to Y$ *be a perfect map from a space* X *having property* wD *onto a regular* T_1*-space* Y. *If there exists a positive natural number* N *such that* $|f^{-1}(y)| \leq N$ *for all* $y \in Y$, *then* Y *has property* wD.

Proof. Let T be a countably infinite discrete set in Y. By Lemma 2.5, it suffices to show that there exists an infinite locally finite family $\mathcal{V}$ of open sets in Y such that $V \cap T \neq \emptyset$ for infinitely many V in $\mathcal{V}$. To prove this, we will use the following well-known fact about perfect maps: If $\mathcal{S}$ is a locally finite collection in X, then $\{f(S): S \in \mathcal{S}\}$ is locally finite in Y. Since f is a closed map it suffices to find a locally finite family $\{U_t: t \in T\}$ of open sets in X such that $f^{-1}(t) \subset U_t$ for infinitely many $t \in T$.

By passing to a subsequence, we may assume that there exists a positive integer $m \leq N$ such that $|f^{-1}(t)| = m$ for all $t \in T$. The proof now proceeds by induction on m. We will do the case $m = 2$, and leave the remainder of the proof (including the case $m = 1$) to the reader. Let $f^{-1}(t) = \{a_t, b_t\}$ for all $t \in T$. Since $\{f^{-1}(t): t \in T\}$ is a discrete family in X, the sets $A = \{a_t: t \in T\}$ and $B = \{b_t: t \in T\}$ are discrete sets in X. By property wD there exist an infinite set $T' \subset T$ and a discrete family $\{V_t: t \in T'\}$ of open sets in X such that $V_t \cap A = \{a_t\}$ for all $t \in T'$. Next we apply property wD to $\{b_t: t \in T'\}$, and get an infinite subset $T'' \subset T'$ and a discrete family $\{W_t: t \in T''\}$ of open sets in X such that $W_t \cap B = \{b_t\}$ for all $t \in T''$. Then $\{V_t \cup W_t: t \in T''\}$ is a locally finite family of open sets in X such that $f^{-1}(t) \subset (V_t \cup W_t)$ for all $t \in T''$. This completes the proof.

5.2. *Proposition. If* $f: X \to Y$ *is a perfect map such that* $f^{-1}(y)$ *is finite for all* $y \in Y$*,* X *is a* T_1*-space having property* D*, and either* X *or* Y *is regular, then* Y *has property* D.

Proof. Since regularity is preserved by perfect maps, it suffices to prove the result for the case in which Y is regular. If T is a countable, discrete set in Y, then since X is a T_1-space, $S = \cup\{f^{-1}(t): t \in T\}$ is a countable, discrete set in X. By property D, there exists a discrete family $\{V_s: s \in S\}$ of open sets in X such that $V_s \cap S = \{s\}$ for all $s \in S$. Put $U_t = \cup\{V_s: s \in f^{-1}(t)\}$. Then $\{\mathrm{Int}_Y(f(U_t)): t \in T\}$ is a locally finite family of mutually disjoint open sets in Y such that $\mathrm{Int}_Y(f(U_t)) \cap T = \{t\}$ for all $t \in T$. Since Y is regular, this completes the proof.

Next we show that properties D and wD are reflected by perfect maps.

5.3. *Proposition. If* $f\colon X \to Y$ *is a (quasi) perfect map from a* T_1*-space* X *onto a space* Y *having property* wD, *then* X *has property* wD.

Proof. Let A be a countably infinite discrete set in X. Since f is a closed map, f(A) is a discrete set in Y. Since $f^{-1}(y)$ is (countably) compact for all $y \in Y$, we see that $f^{-1}(y) \cap A$ is finite for all $y \in Y$. In particular, f(A) is infinite. Since f is continuous, every discrete family of open sets in Y can be brought back by f^{-1} to a discrete family of open sets in X. Thus, there exists a discrete family $\mathcal{V}$ of open sets in X such that $V \cap A$ is non-empty and finite for every $V \in \mathcal{V}$. Since every subset of A is closed in X, we may refine $\mathcal{V}$ to get a discrete family of open sets in X, each of which contains exactly one point of A.

In a similar manner, one can show that property D is reflected by (quasi) perfect maps $f\colon X \to Y$ where X is a Urysohn space.

The remainder of this section is concerned with showing that finite-to-one perfect maps can destroy property wD. In particular, we will show that certain finite-to-one perfect images of the Niemytzki plane, and of the Pixley-Roy space, do not have property wD.

5.4. *Lemma. Let* R *be a closed discrete subset of a* *space* X, *and* $\{F_\alpha: \alpha < k\}$ *a family of mutually disjoint subsets of* R. *Let* p: X $\to$ Y *be the quotient map which collapses each* F_α *to a single point and is one-to-one on* $X \setminus \cup\{F_\alpha: \alpha < k\}$. *Then* p *is a closed map. Further,* p *is perfect if and only if each* F_α *is finite.* (The proof is routine.)

5.5. *Lemma. Let* R *be a closed discrete subset of a Urysohn space* X, *and* $\mathcal{B}$ *a countable family of infinite subsets of* R *having the property*

(*) *for every countable* $H \subset R$, *if* $H \cap B \neq \emptyset$ *for all* $B \in \mathcal{B}$, *then* H *does not have property* D *in* X.

Then there exists a perfect map p: X $\to$ Y *such that* Y *does not have property* wD.

Proof. Let $\mathcal{B} = \{B_i: i < \omega\}$ and let $\{F_i: i < \omega\}$ be a family of mutually disjoint subsets of R such that (1) F_i is finite for all $i < \omega$, and (2) if $j \leq i$ then $F_i \cap B_j \neq \emptyset$. Let p: X $\to$ Y be the quotient map which collapses each F_i to a point. By Lemma 5.4, p is a perfect map, and clearly $\{p(F_i): i < \omega\}$ is a countable discrete set of points in Y. We show that this set does not have property wD in Y. If it did have property wD in Y then there would exist a subsequence $\{p(F_{i_j}): j < \omega\}$ and open sets V_j in Y such that (a) $p(F_{i_j}) \in V_j$ for all $j < \omega$, and (b) $\{V_j: j < \omega\}$ is discrete in Y. Now (b) implies that $\{p^{-1}(V_j): j < \omega\}$ is discrete in X, and (a) implies that F_{i_j} is a finite subset of $p^{-1}(V_j)$. Since X is a Urysohn space, we may put the points of F_{i_j} into open sets which are contained in V_j and have disjoint closures. This shows that $F = \cup\{F_{i_j}: j < \omega\}$ has property D in X. By (2), F intersects each $B \in \mathcal{B}$, and thus

F does not have property D in X. This contradiction completes the proof.

We now give several examples of how Lemma 5.5 may be applied.

5.6. *Example.* Let N be the Niemytzki plane. Recall that N is the set of all points in the upper half plane: $N = \{(x,y): y \geq 0\}$. Let R denote the x-axis: $R = \{(x,0): -\infty < x < \infty\}$.

The topology on N is defined as follows: Points above the x-axis have their usual open disks as basic neighborhoods, and points p on the x-axis have as basic neighborhoods all sets of the form $\{p\} \cup A$ (where A is an open disk in N tangent to the x-axis at p). Let $\mathcal{B}$ be a countable base for the usual topology on R. We show that (*) of Lemma 2.2 holds for R and $\mathcal{B}$. Suppose that H is a countable subset of R such that $H \cap B \neq \emptyset$ for all $B \in \mathcal{B}$. To show that H does not have property D in N, it suffices to show that H and $R \setminus H$ cannot be separated by open sets in N. This is done in the standard way using a Baire category argument.

In a similar manner, Lemma 5.5 can be applied to $S \times S$, where S is the Sorgenfrey line, and to many versions of Niemytzki's space such as the next example.

5.7. *Example.* (R. W. Heath [He]) Let H be Heath's V-space. Recall that the set $H = \{(x,y): y \geq 0\}$ and the topology for H is defined as follows. All points (x,y) with $y > 0$ are isolated, and for a point $p = (p_0, 0)$ on the x-axis, a local base is given by all sets of the form

$$V(p,n) = \{p\} \cup \{(x,y): 0 < y < 1/n, \text{ and } y = \pm((\pi/4)x - (\pi/4)p_0)\}.$$

where $0 < n < \omega$. Thus, $V(p,n)$ is a subset of the union of two lines through p_0 having slopes $\pi/4$ and $-\pi/4$. Clearly, the same choice of R and β as in 6.6 shows that (*) of Lemma 5.5 holds for the space H.

5.8. *Remark.* Heath's V-space H is homeomorphic to a closed subset of the Pixley-Roy space (this also has been noticed by D. Lutzer and H. Bennett). Recall that the Pixley-Roy space Λ consists of the set of all finite subsets of the real line R with the following topology. For each x in Λ and each open set U in the usual topology on R, put $[x,U] = \{y \in \Lambda: x \subset y \subset U\}$. The collection of all such $[x,U]$ forms a base for the topology on the Pixley-Roy space [PR]. We show that H is homeomorphic to $Z = \{x \in \Lambda: |x| \leq 2 \text{ and } x \neq \emptyset\}$. Clearly, Z is closed in Λ. We show that Z is homeomorphic to the copy of H obtained by rotating H by 45° in the plane. Let $H' = \{(x,y): x \leq y\}$ with all points (x,y) with $x < y$ isolated, and for points (x,x) take as a local base all sets of the form

$$V((x,x),n) = \{(x,y): x \leq y < 1/n\} \cup \{(y,x): x \leq y < 1/n\}$$

where $0 < n < \omega$. Clearly H' is homeomorphic to H. Define a map $f: H' \to Z$ by $f((x,x)) = \{x\}$ and $f((x,y)) = \{x,y\}$ if $x < y$. Since f is one-to-one and $f(V((x,x),n)) = [\{x\},(x-1/n, x+1/n)]$, it follows that f is the desired homeomorphism.

5.9. *Example.* The Pixley-Roy space satisfies (*) of

Lemma 5.5. This follows at once from Remark 5.8 because Λ has a closed subspace which satisfies (*) of 5.5. This shows that Λ does not satisfy property D, and this slightly generalizes the well-known results that Λ is not pseudonormal, therefore neither normal nor countably paracompact (cf. [vD_2]). Further, this shows that there exists a space Y which is a perfect finite-to-one image of Λ, which does not have property wD. The space Λ is a hereditarily metacompact Moore space (cf. [vD_2]), and both these properties are preserved by perfect maps [Wo_1], [Wo_2]. Thus Y is a hereditarily metacompact Moore space (having the countable chain condition) which does not have property wD.

6. Some New Uses of Property wD

In this section we show how Example 5.6 and similar examples can be used concerning how perfect maps destroy submetrizability and realcompactness.

Recall that a $T_{3\ 1/2}$-space is called *realcompact* if it is homeomorphic to a closed subset of a product of copies of the real line. It is easy to see that every product of real lines has property wD, and hence every realcompact space has property wD (Corollary 4.4b). The fact that realcompact spaces have property wD has been noted earlier in [SA] and [vD_3], and it also follows from basic facts about realcompactness and Morita's characterization of property wD given in [Mo_1]. Recall again that a space X with topology T is called *submetrizable* provided that there exists a metrizable topology M on X such that $M \subset T$. As we have mentioned, every regular, submetrizable space has property wD (Corollary 3.5). It is known that submetrizable $T_{3\ 1/2}$-spaces of cardinality c have the following three properties: (i) they are realcompact [GJ, 8.17 and 15.24], (ii) they have a G_δ-diagonal (cf. [BL]), and (iii) they have property wD (for two reasons).

We now show that perfect maps can destroy submetrizability by destroying any one of these three properties.

6.1. A perfect map defined on a submetrizable space which destroys the G_δ-diagonal, but preserves realcompactness and property wD. F. G. Slaughter, Jr. [S] and V. Popov [P] have shown that there exists a perfect (2-to-1) map defined on the disjoint union of two copies of the Michael line whose image does not have a G_δ-diagonal. Since the image is a paracompact space of cardinality c, it is realcompact [K] and has property wD.

6.2. A perfect map defined on a submetrizable space which destroys realcompactness, but preserves the G_δ-diagonal and property wD. S. Mrówka [Mr_1], [Mr_2] has constructed a (2-to-1) perfect map, defined on the disjoint union of two copies of the Niemytzki plane, whose image is not realcompact. This image does have a G_δ-diagonal since both the domain and range are Moore spaces [Wo_2], and the image has property wD by Proposition 5.1.

6.3. A perfect map defined on a submetrizable space which destroys property wD (and therefore realcompactness) but preserves the G_δ-diagonal. Example 5.6 in this paper shows that there exists a (finite-to-one) perfect map whose domain is the Niemytzki plane and whose image does not have property wD. The image has a G_δ-diagonal, and of course is not realcompact.

In order to answer several questions of S. Mrówka, Akio Kato [Ka_1] constructed examples of first countable spaces which show that

6.4. A space which is the union of a countable, closed discrete set and a realcompact set need not be realcompact, and

6.5. There is a finite-to-one perfect map f: $X \to Y$ which destroys realcompactness and such that $|\{y \in Y: |f^{-1}(y)| \geq 2\}| = \omega$.

The examples constructed in §5, can be used to show that both 6.4 and 6.5 hold (Eric van Douwen has informed me that R. Pol has also constructed a simple example to show that 6.4 obtains, but Pol's example is not first countable). Let f: $N \to Y$ be the finite-to-one perfect map constructed on the Niemytzki plane N as in Lemma 6.5. The map f destroys realcompactness because it destroys property wD. Further, there are only countably many $y \in Y$ such that $|f^{-1}(y)| \geq 2$. This shows that 6.5 obtains. To see that 6.4 obtains use the image space Y. Let $A = \{y \in Y: |f^{-1}(y)| \geq 2\}$ and $B = Y\setminus A$. Then A is a countable, closed discrete subspace of Y (Lemma 5.4) and B is realcompact because it is homeomorphic to a subspace of Niemytzki's space N. Thus $Y = A \cup B$, but Y is not realcompact.

To get analogous examples for 6.4 and 6.5 concerning N-compactness instead of realcompactness, we use (instead of the Niemytzki plane) Heath's V-space (see Example 6.7) or Mrówka's N-compact version of Niemytzki's space [Mr_2].

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