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1. Introduction

This paper shows how the concept of θ -space and other related conditions can be used to obtain monotonic uniformizations fundamental in base of countable order theory. From these results a number of sharp metrization and developability theorems of the factorization type are obtained. The transitions from θ -conditions to monotonic ones are accomplished via the concepts of monotonic β -space [C] and monotonic semi-stratification [C] which play a basic role here. A typical result is that a regular T_1 monotonic β -space has a base of countable order if and only if it is a θ -space. Another aspect emphasized is that many important results involving G_δ -diagonal remain valid when that concept is replaced by the much weaker one of θ -diagonal. Thus many of the results clarify the nature of certain classical ones by elimination of superfluous hypotheses. This is pointed out in several places in Section 5 and 6 without attempting to be exhaustive.

In [C] the concept of an (m) -sequence of ordered covers was combined with monotonic β -space to obtain characterization theorems, e.g., of base of countable order. One of the main objects here is to point out that weaker θ -conditions (defined in Section 2) can be used instead of ordered covers. It is of interest that the notion of (m) -sequence of ordered covers [C] which is equivalent to primitive (m) -sequence can

be replaced by the weaker notion of star-(m) function (= a θ -type condition), since it represents a departure from the method of primitive sequences which forms the basis of the approach to base of countable order theory initiated in [WoW] and carried out in subsequent publications (see references in [WW₂]).

The use of θ -space concepts permits obtaining analogues of results in developable space theory. For example, the factorization of regular T_1 spaces having bases of countable order into monotonically semistratifiable $w\theta$ -spaces (Theorem 5.5) is an analogue of the theorem [Ho₁] that Moore spaces are equivalent to regular T_1 semi-stratifiable $w\theta$ -spaces.

In section 2, a unified way of considering various kinds of uniformizations of topological concepts such as first countability is expounded. In section 3 Chaber's concepts are presented. The main theorem is proved in section 4, and the paper concludes in sections 5 and 6 with numerous theorems based in part, at least, on the main theorem.

This paper relates to some of the work in [C] and [FL], the reading of which was instructive.

2. Uniformization of Topological Properties

This section expounds a unifying view point for considering different types of structures on a space and it contains definitions and terminology used later. We consider various methods of uniformizing topological properties (m) of sequences. For convenience of exposition we focus on 3 such properties labelled (d), (q), and (Δ) (cf. [C, Section 2]), but the same procedures may be used for other properties (m).

The properties are defined for sequences $\langle B_n : n \in \mathbb{N} \rangle$ of subsets of a space X :

(d) If $x \in \bigcap \{B_n : n \in \mathbb{N}\}$, $\{B_n : n \in \mathbb{N}\}$ is a base at x .

(Δ) If $x \in \bigcap \{B_n : n \in \mathbb{N}\}$, then $\bigcap \{B_n : n \in \mathbb{N}\} = \{x\}$.

(q) If $x \in \bigcap \{B_n : n \in \mathbb{N}\}$ and for all $n \in \mathbb{N}$,

$y_n \in \bigcap \{B_k : 1 \leq k \leq n\}$, then $\langle y_n : n \in \mathbb{N} \rangle$ clusters.

The properties listed correspond respectively to first countability, points being G_δ 's, and q -space. For the first two ways of uniformizing (m), let (X, τ) be a space and $Y \subseteq X$ and let $\mathcal{G} = \langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ be a sequence of bases of Y in X ; i.e. each $\mathcal{G}_n \subseteq \tau$ and if $y \in Y$ and $U \in \tau$, there exists $G \in \mathcal{G}_n$ such that $y \in G \subseteq U$.

2.1. *Uniformization of developable type.* To obtain this type, suppose for all representatives G of \mathcal{G} (i.e. $G_n \in \mathcal{G}_n$ for all $n \in \mathbb{N}$), if $y \in Y \cap \bigcap \{G_n : n \in \mathbb{N}\}$, then require that $\langle G_n : n \in \mathbb{N} \rangle$ has property (m) at y (and in Y , for (q)). In particular, if $Y = X$, then

(a) If (m) = (d), X is *developable*.

(b) If (m) = (Δ), X has a G_δ -*diagonal*.

(c) If (m) = (q), X is *quasi-complete* [Cr].

(Note that open covers \mathcal{G}_n may be used here instead of bases.)

2.2. *Uniformization of monotone type.* Here the only change from 2.1 is to require that (m) is required to hold only for decreasing representatives G of \mathcal{G} , (i.e. $G_{n+1} \subseteq G_n$ for all $n \in \mathbb{N}$) and $y \in Y$. If $Y = X$, then:

(a) If (m) = (d), X has a *base of countable order*

[A, WoW].

- (b) If $(m) = (\Delta)$, X has *diagonal a set of interior condensation* $[W_2]$.
- (c) If $(m) = (q)$, we say that X is a *monotonic q -space* (see Section 5 for further remarks).

2.3. *Uniformization of primitive type.* For each $n \in \mathbb{N}$, let \mathcal{W}_n be a well-ordered collection of open sets in X such that \mathcal{W}_n covers Y . For each $x \in Y$, let $F(\{x\}, \mathcal{W}_n)$ denote the first element of \mathcal{W}_n that contains x . Then we require that the sequence $\langle F(\{x\}, \mathcal{W}_n) : n \in \mathbb{N} \rangle$ have (m) for all $x \in Y$. If $Y = X$, then:

- (a) If $(m) = (d)$, X has a *primitive base* $[WW_2]$.
- (b) If $(m) = (\Delta)$, X has a *primitive diagonal* $[WW_2]$.
- (c) If $(m) = (q)$, X is a *primitive q -space*.

(Case (c) is called *primitively quasi-complete* in $[WW_2]$.)

2.4. *Uniformization of star functional type.* This involves a function $h: \mathbb{N} \times Y \rightarrow \tau$, where τ is the topology on X , and for all $x \in Y$ and $n \in \mathbb{N}$, $x \in h(n+1, x) \subseteq h(n, x)$. Define for each $x \in X$ and $n \in \mathbb{N}$, $h^*(n, x) = \cup\{h(n, z) : z \in h(n, x) \text{ and } x \in h(n, z)\}$. Then require for all $y \in Y$ that the sequence $\langle h^*(n, y) : n \in \mathbb{N} \rangle$ has property (m) at y . If $Y = X$, then:

- (a) If $(m) = (d)$, X is a θ -*space* $[Ho_2]$.
- (b) If $(m) = (\Delta)$, X has a θ -*diagonal*.
- (c) If $(m) = (q)$, X is a $w\theta$ -*space* $[Ho_2]$.

2.5. *Proposition.* The uniformizations above satisfy $2.1 \rightarrow 2.2 \rightarrow 2.3 \rightarrow 2.4$, for a fixed (m) .

2.6. *Notes.* In a number of cases above, especially in 2.4, these are not the original definitions of the concepts.

The functional approach has been effectively used by Heath [H] and Hodel [Ho_2]. The recent paper of [FL] continues this work and was a stimulus for the formulation of 2.4 above. The condition called θ -diagonal is equivalent to Condition II of [FL]. The uniformizations in 2.2 and 2.3 arose from the work in [WoW] and [WW_2]. However in those works the concept of primitive sequence is used rather than ordered covers. The uniformization of 2.3 could be phrased in terms of primitive sequences. The original definition of base of countable order given in [A] was not of this uniformization type. The developable type formulation is by now classical. That all of these can be looked at as different types of uniformization of simple properties seems worth emphasizing. Note also that one could add to the above list a quasi-developable type of uniformization in an obvious way.

2.7. *Terminology.* If (m) is a property of sequences, and h is a function as in 2.4, h will be called a *star-(m) function* of Y in X , or if $Y = X$, just a *star-(m) function*.

A sequence of bases as in 2.2 will be called a *monotonic (m)-sequence of bases* of Y in X , or if $Y = X$, a *monotonic (m)-sequence of bases*.

A sequence of ordered covers as in 2.3 is called an *(m)-sequence of ordered covers* in [C].

3. Monotonic β -Spaces

In [C] monotonic formulations of β -space [Ho_2] and semi-stratifiable space [Cr] are defined and used to obtain a unified approach to certain characterization theorems. In [C] these concepts are combined with certain sequences of

ordered covers (equivalently, with primitive sequences) or with point-countability weak covering conditions. In the sequel it is shown that in certain theorems (m)-sequences of ordered covers can be replaced by the weaker property of star-(m) function, where (m) is a monotone property (see Section 4).

3.1. *Definition [C].* A space X is called a *monotonic β -space* if and only if for all $x \in X$ there is a decreasing sequence $\langle B_n(x) : n \in \mathbb{N} \rangle$ of bases at x such that if, for all $n \in \mathbb{N}$, $B_n \in \beta_n(x_n)$, $B_{n+1} \subseteq B_n$, and $\cap \{B_n : n \in \mathbb{N}\} \neq \emptyset$, then $\langle x_n : n \in \mathbb{N} \rangle$ has a cluster point.

If, in addition, $y \in \cap \{B_n : n \in \mathbb{N}\}$ implies that $\langle x_n : n \in \mathbb{N} \rangle$ clusters to y , then X is called *monotonically semi-stratifiable*.

Chaber points out that if in the above it is not assumed that the sequences $\langle B_n : n \in \mathbb{N} \rangle$ involved in the condition are decreasing, then characterizations of β -space and semi-stratifiable space are obtained. It may be noted that in the definition of *monotonically semi-stratifiable* it is equivalent to require that $\langle x_n : n \in \mathbb{N} \rangle$ converges to y .

In [FL] a space (X, τ) is defined to be a *quasi- β -space* provided that there is a function $g: \mathbb{N} \times X \rightarrow \tau$ such that if $\{x, x_n\} \subset \cap \{g(i, x_i) : 1 \leq i \leq n\}$ for each $n \in \mathbb{N}$, then $\langle x_n : n \in \mathbb{N} \rangle$ has a cluster point.

3.2. *Proposition.* *Every quasi- β -space is a monotonic β -space.*

Proof. Let g be a quasi- β function for (X, τ) . For each $n \in \mathbb{N}$ and $x \in X$, define $\beta_n(x) = \{B \in \tau : x \in B \subseteq \cap \{g(i, x) :$

$1 \leq i \leq n$ }. Clearly $\langle \beta_n(x) : n \in \mathbb{N} \rangle$ is a decreasing sequence of bases at x . If $y \in B_{n+1} \subseteq B_n \in \beta_n(x_n)$ for all $n \in \mathbb{N}$, then $\{y, x_n\} \subseteq \cap \{g(i, x_i) : 1 \leq i \leq n\}$. Hence $\langle x_n : n \in \mathbb{N} \rangle$ clusters. Thus monotonic β -space generalizes β -space and quasi-complete space. The results of Section 5 show that quasi- β space may be replaced by monotonic β -space in Props. 4.2, 4.3, and 4.4 of [FL].

Using Definition 3.1, it may be seen that the properties of monotonic β -space and monotonic semi-stratifiability are countably productive. Using productivity it may be shown that for T_2 spaces, monotonic semi-stratifiability implies diagonal a set of interior condensation (i.e. in the terminology of 2.2, the space has a monotonic (Δ) -sequence of bases). This fact is stated in [C, Prop. 1.8] in different terminology.

3.3. *Terminology.* If X is a space and $Y \subseteq X$, then $\langle \langle \beta_n(x) : n \in \mathbb{N} \rangle : x \in Y \rangle$, where the $\beta_n(x)$'s satisfy the first part of 3.1, is called a *monotonic β -system of Y in X* . If the $\beta_n(x)$'s also satisfy the second part of 3.1 then the system is called a *monotonic semi-stratification of Y in X* .

A number of relationships among the concepts defined here and other well known notions are summarized in diagrams appearing in Section 6.

4. The Main Theorem

In [CCN], if \mathcal{M} and \mathcal{N} are families of subsets of a space X , $\delta\mathcal{M} < \mathcal{N}$ means that each element of \mathcal{N} includes an element

of \mathcal{M} . Also a property (m) of sequences is called *monotonic* if $\langle H_n : n \in \mathbb{N} \rangle$ has (m) and $\delta \langle W_n : n \in \mathbb{N} \rangle < \langle H_n : n \in \mathbb{N} \rangle$ implies that $\langle W_n : n \in \mathbb{N} \rangle$ has (m). A monotonic property (m) is *non-complete* if all sequences H such that $\bigcap \{H_n : n \in \mathbb{N}\} = \emptyset$ have (m). Note that the properties (d), (q), and (Δ) are such properties. The following theorem is fundamental for most of the results of the paper. The terminology (m)-sieve is from [CCN] where the equivalence of (a) and (c) is proved.

4.1. *Theorem.* Suppose (X, τ) is a regular T_1 space, $Y \subseteq X$ has a monotonic β -system in X , and (m) is a noncomplete monotonic property. Then the following are equivalent:

- (a) Y has a monotonic (m)-sequence of bases in X .
- (b) Y has a star-(m) function in X .
- (c) Y has an (m)-sieve in X .

Proof. If (a) holds, Y has a sequence \mathcal{G} of bases in X such that decreasing representatives G of \mathcal{G} have (m) if $Y \cap \{G_n : n \in \mathbb{N}\} \neq \emptyset$. From \mathcal{G} we may derive a primitive (m)-sequence \mathcal{H} of Y in X [WW₂], i.e. a sequence $\langle \mathcal{H}_n : n \in \mathbb{N} \rangle$ of well ordered collections of open sets of X covering Y such that for all $n \in \mathbb{N}$, if $F(A, \mathcal{H}_n)$ denotes the first element of \mathcal{H}_n that includes A , then for each $H \in \mathcal{H}_n$ there is an $x \in Y$ such that: (1) $H = F(\{x\}, \mathcal{H}_n)$, (2) if $x \in Y$, $F(\{x\}, \mathcal{H}_{n+1}) \subseteq F(\{x\}, \mathcal{H}_n)$, and (3) if $\langle H_n : n \in \mathbb{N} \rangle$ satisfies $H_n = F(H_{n+1}, \mathcal{H}_n)$ for all $n \in \mathbb{N}$, then $\langle H_n : n \in \mathbb{N} \rangle$ has (m) in Y . For each $x \in Y$, define $h(n, x) = F(\{x\}, \mathcal{H}_n)$. Then $x \in h(n+1, x) \subseteq h(n, x)$ and $h^*(n, x) = \bigcup \{h(n, y) : y \in h(n, x) \text{ and } x \in h(n, y)\} = h(n, x)$. For each $x \in Y$, $\{h^*(n, x) : n \in \mathbb{N}\}$ has (m) since $h(n, x) = F(h(n+1, x), \mathcal{H}_n)$. (Note that monotonic β -system is not used

in this direction.)

Let $\langle \langle \beta_n(x) : n \in \mathbb{N} \rangle : x \in Y \rangle$ be a monotonic β -system of Y in X and let $h: \mathbb{N} \times X \rightarrow \tau$ be a star-(m) function of Y in X . For each $n \in \mathbb{N}$, let $\mathcal{G}_n = \{B \in \beta_n(x) : x \in Y \text{ and } \bar{B} \subseteq h(n, x)\}$. Then each \mathcal{G}_n is a base (in X) for all points of Y . Suppose $y \in Y$ and $y \in G_{n+1} \subseteq G_n \in \mathcal{G}_n$ for all $n \in \mathbb{N}$. Each $G_n \in \beta_n(x_n)$ for some $x_n \in Y$. Thus $\langle x_n : n \in \mathbb{N} \rangle$ clusters to some $z \in Y$. Since each $x_n \in G_n$, $z \in \overline{\{x_k : k \geq n\}} \subseteq \bar{G}_n \subseteq h(n, x_n)$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exists $j_n \geq n$ such that $x_{j_n} \in h(n, z)$. Since $h(j_n, x_{j_n}) \subseteq h(n, x_{j_n})$, $z \in h(n, x_{j_n})$. Thus $G_{j_n} \subseteq h(j_n, x_{j_n}) \subseteq h^*(n, z)$ for all $n \in \mathbb{N}$. Hence $\delta \langle G_n : n \in \mathbb{N} \rangle < \langle h^*(n, z) : n \in \mathbb{N} \rangle$, so $\langle G_n : n \in \mathbb{N} \rangle$ has (m).

4.2. *Theorem.* If X is a regular T_1 monotonic β -space and (m) is a non-complete monotonic property, then the following are equivalent:

- (a) X has a monotonic (m)-sequence.
- (b) X has a star-(m) function.
- (c) X has an (m)-sieve.

Proof. Put $Y = X$ in 4.1.

5. Applications

We present some new characterizations of various kinds of spaces and obtain some well known theorems as corollaries.

5.1. *Theorem.* Let X be a regular T_1 monotonic β -space. Then the following are equivalent:

- (a) X has a base of countable order.
- (b) X is a θ -space.

Proof. That (a) implies (b) follows from the facts that base of countable order implies primitive base [WW₂, Th. 3.1] and primitive base implies θ -space [FL, Prop. 3.3]. As pointed out in Section 2, X is a θ -space provided X has a star-(d) function. By 4.2, X has a monotonic (d)-sequence of bases and thus X has a base of countable order [WoW, Th. 2].

5.2. *Theorem.* A regular T_1 monotonic β -space is monotonically semi-stratifiable if and only if it has a θ -diagonal.

Proof. A T_2 monotonically semi-stratifiable space X has diagonal a set of interior condensation as pointed out in section 3. Thus X has a monotonic (Δ) -sequence. From this a primitive (Δ) -sequence $\langle \beta_n : n \in \mathbb{N} \rangle$ may be obtained [WW₂, 2.10]. By defining $h(n, x) = F(\{x\}, \beta_n)$ (see proof of 4.1) a star- (Δ) function is obtained. Thus X has a θ -diagonal.

Suppose $\langle \langle \beta'_n(x) : n \in \mathbb{N} \rangle : x \in X \rangle$ is a monotonic β -system for X and h is a star- (Δ) function. For each $n \in \mathbb{N}$ and $x \in X$, let $\beta_n(x) = \{B \in \beta'_n(x) : \bar{B} \subseteq h(n, x)\}$. Then $\beta_n(x)$ is a base at x and $\beta_n(x) \subseteq \beta'_n(x)$. Suppose $y \in B_{n+1} \subseteq B_n \in \beta_n(x_n)$ for all $n \in \mathbb{N}$. By the argument of Theorem 4.1, there is a z such that $B_{j_n} \subseteq h^*(n, z)$ for all $n \in \mathbb{N}$. Thus $\bigcap \{B_n : n \in \mathbb{N}\} = \{z\} = \{y\}$. Hence $\langle x_n : n \in \mathbb{N} \rangle$ clusters to y .

5.3. *Corollary.* A regular monotonic β -space has diagonal a set of interior condensation if and only if it has a θ -diagonal.

5.4. *Remark.* Theorem 5.1 (respectively 5.2) shows that in Cor. 2.4 (respectively, Cor. 2.5) of [C], the concept of (d)-sequence (respectively, (Δ) -sequence) of ordered covers

may be replaced by θ -space (respectively, θ -diagonal) and Cor. 5.3 above shows that in Prop. 1.8 of [C], W_δ -diagonal (= diagonal a set of interior condensation) may be replaced by θ -diagonal. A similar remark holds for (p)-sequence of ordered covers (see [C, page 211] for (p)) and star-(p) function in Cor. 2.6 of [C].

5.5. *Theorem.* *A regular T_1 space has a base of countable order if and only if it is a monotonically semi-stratifiable $w\theta$ -space.*

Proof. Let X be a regular monotonically semi-stratifiable $w\theta$ -space. Proceed as in 5.2 to get a primitive (Δ) -sequence $\langle \beta_n : n \in \mathbb{N} \rangle$. Using regularity and [WW₂, 2.10] we can require of this sequence that if $\langle B_n : n \in \mathbb{N} \rangle$ is any sequence such that each $B_n = F(B_{n+1}, \beta_n)$ (see 4.1 for notation), then each $\overline{B_{n+1}} \subseteq B_n$. Define $h(n, x) = F(\{x\}, \beta_n)$ to get a star- (Δ) function. Let $K \subseteq X$ be compact and $x \in X$. If each $h(n, x) \cap K \neq \emptyset$, then there exists $y \in \bigcap \{\overline{h(n, x)} : n \in \mathbb{N}\} \cap K$. Since $\overline{h(n+1, x)} \subseteq h(n, x)$, $y \in \bigcap \{h(n, x) : n \in \mathbb{N}\} = \{x\}$. Thus $x \in K$. Hence X satisfies Condition I of [FL]. By [FL; Cor. to 2.2], X is a θ -space. Thus sufficiency follows from 5.1. There is a straight-forward proof of necessity.

5.6. *Remark.* Theorem 5.5 is an analogue of Hodel's result [Ho₁, Cor. 4.6]: A regular T_1 space is a Moore space if and only if it is a semi-stratifiable $w\theta$ -space. Hodel's theorem is a corollary of the following corollary of 5.6.

5.7. *Corollary.* *A regular T_1 space is a Moore space if and only if it is a θ -refinable monotonically*

semi-stratifiable $w\theta$ -space (equivalently, a θ -refinable monotonic β -space which is a θ -space).

Since every quasi-developable space is a θ -space the result of [BB, Cor. 3.2] that every regular T_1 θ -refinable β -space with a quasi-development is a Moore space follows from the last statement of 5.7.

In [WW₃, Th. 4.1] the following is proved:

5.8. *Theorem.* *An essentially T_1 space has a base of countable order if and only if it has a primitive base and closed sets are sets of interior condensation.*

Recall that a subset M of a space X is called a *set of interior condensation* [WW₁] if there exists a sequence $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ of bases for M in X such that if $G_{n+1} \subseteq G_n \in \mathcal{G}_n$ for all $n \in \mathbb{N}$, then $\bigcap \{G_n : n \in \mathbb{N}\} \subseteq M$. Note that a space X has diagonal a set of interior condensation if its diagonal in $X \times X$ is such a set.

In [C], Chaber points out that in the regular case one can replace the condition on closed sets in 5.8 by the condition of being a monotonic β -space. This is explained by Theorem 5.1 since a space with a primitive base is a θ -space. Another way of viewing Chaber's result is via the following.

5.9. *Proposition.* *In a regular monotonic β -space having a θ -diagonal closed sets are sets of interior condensation.*

Proof. Suppose M is closed in a regular monotonic β -space X having a θ -diagonal. Then M has a monotonic β -system in X and a star- (Δ) function in X . By Theorem 4.1, M has a monotonic (Δ) -system $\mathcal{G} = \langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ of bases in X .

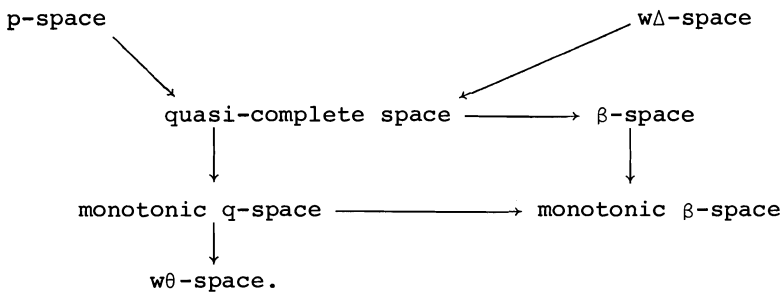
If $\langle G_n : n \in \mathbb{N} \rangle$ is a decreasing representative of \mathcal{G} (i.e. $G_{n+1} \subseteq G_n \in \mathcal{G}_n$) then $\cap\{G_n : n \in \mathbb{N}\}$ is either empty or is $\{x\}$ for some $x \in M$.

5.10. *Note.* Another way of seeing 5.9 is to use Theorem 5.2 and the fact stated (without proof) in [C, Prop. 1.4] that closed sets are sets of interior condensation in a regular monotonically semi-stratifiable space.

5.11. *Corollary.* If X is a regular θ -refinable monotonic β -space with a θ -diagonal then closed sets are G_δ -sets in X and X has a G_δ -diagonal.

Proof. In a θ -refinable space closed sets of interior condensation are G_δ -sets [WW₄, Th. 5.6]. Also by 5.3, X has diagonal a set of interior condensation and by [WW₄, Th. 5.7], X has a G_δ -diagonal.

Monotonic q -spaces are a not necessarily first countable analogue of bases of countable order. The condition called β_c in [W₁, Def. 2.3] implies monotonic q -space and for regular spaces they are equivalent. Thus in the regular case monotonic g -spaces are the uniformly λ -complete open continuous images of regular T_1 M -spaces [W₁, Th. 4.2]. The following diagram points out some relationships:



5.12. *Theorem.* A regular T_1 monotonic β -space is a monotonic q -space if and only if it is a $w\theta$ -space.

Proof. The necessity is clear and the sufficiency follows from Theorem 4.1.

5.13. *Theorem.* A regular T_1 space has a base of countable order if and only if it is a monotonic q -space with a θ -diagonal.

Proof. This follows from Theorems 5.2, 5.5 and the above diagram.

Theorems 3.1 and 3.3 of [BB] are immediate corollaries of this theorem, the relationships in the diagram above, and the observation that a T_1 quasi-developable space has a θ -diagonal.

5.14. *Corollary [W₂].* A regular T_1 space has a base of countable order if and only if it is a monotonic q -space with diagonal a set of interior condensation.

6. Metrization and Summary

From the preceding results and a theorem of Arhangel'skii [A, Th. 2] a number of metrization theorems may be obtained.

6.1. *Theorem.* Let X be a T_2 paracompact space. Then the following are equivalent:

- (a) X is metrizable.
- (b) X is a monotonic β -space and a θ -space.
- (c) X is a monotonic q -space with a θ -diagonal.
- (d) X is monotonically semi-stratifiable with a θ -diagonal.

Proof. Arhangel'skii's theorem [A, Th. 2] states that a

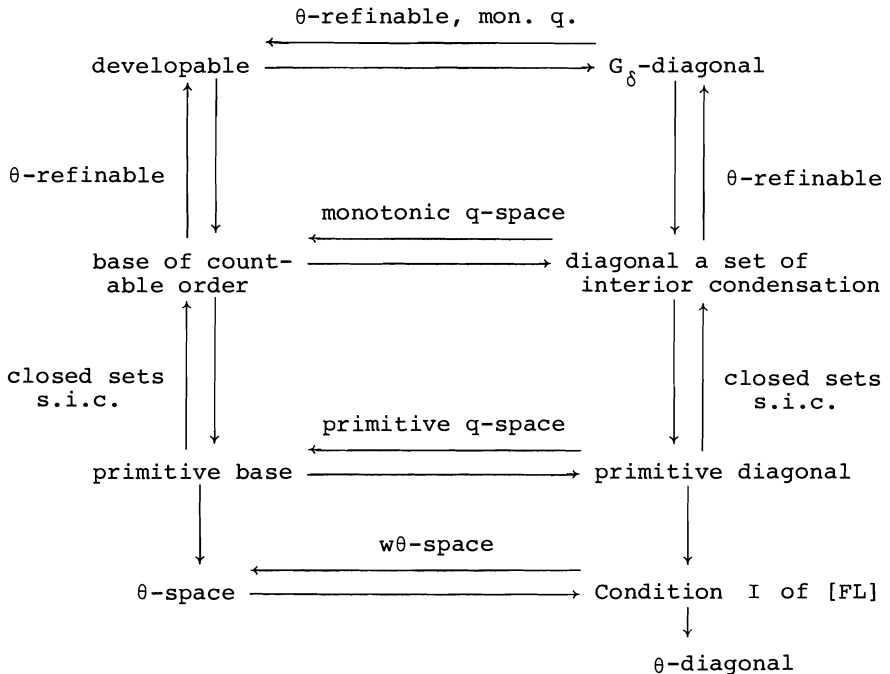
T_2 paracompact space having a base of countable order is metrizable. Thus the results follow from Theorems 5.1, 5.13, and 5.2.

6.2. *Corollary [Bo,O]. A paracompact p-space with a G_δ -diagonal is metrizable.*

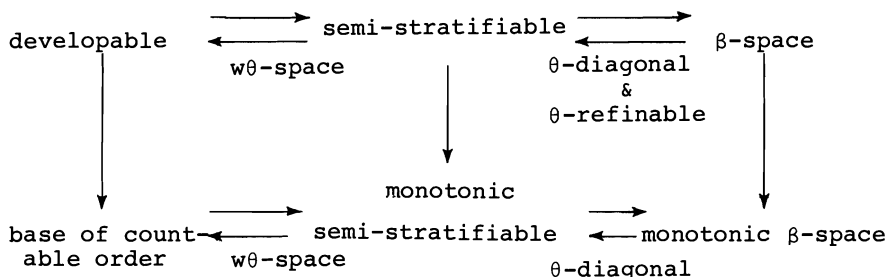
6.3. *Corollary [FL]. A paracompact p-space with a θ -diagonal is metrizable.*

For regular T_1 spaces relationships among various kinds of diagonals are summarized below. The arrows denote implication and if additional hypotheses are needed they are written adjacent to the arrows.

6.4. *Diagram.*



6.5. *Diagram.*



In conclusion, we discuss a theorem which has a number of interesting consequences.

6.6. *Theorem [FL, Lemma 3.4]. Every regular θ -refinable β -space with a θ -diagonal has a θ -separating open cover.*

Definitions of β -space and θ -separating open cover may be found in [Ho₁]. The proof in [FL] is closely related to the proof in [Ho₃, Th. 3.2] of a related theorem. A similar technique involving θ -refinability and König's lemma was used in [WoW, Th. 3]. We prove the following generalization more briefly using some known results.

6.7. *Theorem. Every regular θ -refinable monotonic β -space with a θ -diagonal has a θ -separating open cover.*

Proof. By Corollary 5.11, such a space X has a G_δ -diagonal. Let $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ be a sequence of bases defining the G_δ -diagonal. By [BL, Lemma 4], for each $n \in \mathbb{N}$ there is an open refinement $\cup \{V_{nm} : m \in \mathbb{N}\}$ of \mathcal{G}_n such that for all $x \in X$ some V_{nm} has exactly one element containing x . Suppose $x, y \in X$ and $x \neq y$. There exists n such that $y \notin \text{st}(x, \mathcal{G}_n)$.

There also is an m such that x is in only one element V of \mathcal{V}_{nm} . Since $V \subseteq \text{st}(x, \mathcal{G}_n)$, $y \notin V$. Hence $\cup\{\mathcal{V}_{nm} : n, m \in \mathbb{N}\}$ is a θ -separating open cover of X .

6.8. *Remark.* The referee observed that 6.7 follows from 5.1 and [Ho₃, 3.3]. The above proof illustrates a different approach.

6.9. *Theorem.* Every hereditarily weakly θ -refinable monotonic β -space with a θ -diagonal has a θ -separating open cover and a quasi- G_δ -diagonal.

Proof. A space X satisfying the hypothesis has diagonal a set of interior condensation. Using a monotonic (Δ) -sequence of bases, in place of a (d) -sequence in the proof of [BB, Th. 3.4], a quasi- G_δ -diagonal sequence $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ may be obtained. The argument of the proof of 6.7 may be applied to this sequence to obtain a θ -separating open cover.

6.10. *Questions.*

1. Is every monotonically semi-stratifiable hereditarily θ -refinable space semi-stratifiable?

2. Does every primitive q -space with a θ -diagonal have a primitive base?

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