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Research Announcement:

INCOMPRESSIBLE SPACES

by

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INCOMPRESSIBLE SPACES

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P. Fletcher calls a space *incompressible* if it admits no homeomorphism onto a proper subset of itself. All closed n-manifolds are incompressible for example, while most AR's are compressible.

I. Unions

In [F,S], Fletcher and Sawyer prove that the disjoint union of two connected incompressible spaces is again incompressible, and they ask if "connected" can be removed. We show in Theorem 1 that it cannot be removed, and we generalize their result in Theorem 2. Theorem 3 demonstrates the importance of the union being disjoint in their hypothesis. All proofs are given in [F,H].

Theorem 1. *There exist two incompressible compact metric spaces whose disjoint union is compressible.*

Theorem 2. *If, in each of* X *and* Y, *the set of components is discrete, then* X *and* Y *incompressible implies that their disjoint union is incompressible.*

Theorem 3. *There exist two incompressible metric continua that intersect in exactly one point whose union is compressible.*

II. Products

Fletcher and Sawyer ask, also in [F,S], whether products preserve incompressibility. That question has been answered by L. S. Husch.

In [H] Husch establishes the existence of a three dimensional incompressible metric space whose product with S' is compressible by an ingenious argument using McMillan's uncountable collection of open contractible subsets of S^3 no two of which are homeomorphic, [M]. In Theorem 4 we construct two relatively simple one-dimensional incompressible metric continua whose product is compressible. Infinite products are usually compressible however, as Theorem 5 demonstrates.

Theorem 4. *There exist two* 1-*dimensional incompressible metric continua whose product is compressible.*

Theorem 5. *If, for each* n, X_n *is a separable metric space that contains an arc, then* $\Pi_{n=1}^{\infty} X_n$ *is compressible.*

Question 1. *Does there exist an incompressible infinite product (where each factor is non-degenerate)?*

Note 1. There does exist an infinite-dimensional incompressible metric continuum. Namely, let $X = S_1 \cup S_2 \cup \cdots \cup \{P\}$ where (i) $S_i \cap S_j = \emptyset$ if $|j-1| > 1$, (ii) $S_i \cap S_{i+1}$ is a single point for all i, (iii) $\{S_i\}_{i=1}^{\infty} \to p$, and (iv) S_{2^i} is an i-dimensional sphere and each other S_j is a simple closed curve.

III. Incompressible AR's

Clearly, there exist incompressible ANR's of every finite dimension. In [F,S], the authors observed that the "Dunce's cap" is a 2-dimensional incompressible AR and, as pointed out by W. R. R. Transue, so is the "House With Two Rooms." This latter example appears to generalize to all finite dimensions, and then, using these sets in place of the spheres in Note 1 above, an infinite dimensional incompressible AR could be constructed. Surprisingly, there also exists a one dimensional incompressible AR;

Question 2. *Do there exist two incompressible (compact metric)* AR's *whose product is compressible?*

Note 2. It follows from Theorem 5, that an infinite product of (non-degenerate) AR's must be compressible.

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