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## Research Announcement:

### PLANE INDECOMPOSABLE CONTINUA, PRIME ENDS, AND EMBEDDINGS OF THE PSEUDO-ARC

by

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**PLANE INDECOMPOSABLE CONTINUA,  
PRIME ENDS, AND EMBEDDINGS  
OF THE PSEUDO-ARC**

**Michel Smith**

**I. Introduction**

Beverly Brechner [2] has an example of an embedding of a pseudo-arc  $M$  in the plane which she conjectures has the property that no component of  $M$  is arcwise accessible from the complement of  $M$  at more than one point.

Here the theorems used to construct such an embedding are stated. Suppose that a continuum  $X$  is said to have property IM if  $X$  is a plane continuum and no component of  $X$  is arcwise accessible at more than one point from the complement of  $X$ . A technique for producing plane indecomposable continua with property IM is described. A continuum which has property IM is a plane continuum each of whose prime ends has the whole continuum as its impression, and hence property IM is of interest in prime end theory. Beverly Brechner [2] has pointed out that the Knaster embedding of the pseudo arc in the plane [3] does not have property IM. The embedding of the pseudo-arc in the plane with property IM is used to produce more embeddings of the pseudo arc in the plane. Specifically there are  $c$  inequivalent embeddings of the pseudo arc in the plane, which answers a question posed by Beverly Brechner [2]. In particular there are pseudo-arcs exactly  $N$  components of which are accessible at more than one point and countably infinitely many components of which are accessible at more

than one point. The author wishes to indicate that these results were independently discovered by Wayne Lewis. The aim of this paper is to announce results that the author has submitted elsewhere.

Definitions and Notations: Space is assumed to be the plane, denoted by  $E^2$ , with the standard Euclidean distance  $d$ . By a disc is meant a homeomorphic copy of the square disc  $[0,1] \times [0,1]$ . If  $H$  is a set then  $\text{Int}(H)$  denotes the interior of  $H$  and  $\text{Bd}(H)$  denotes the boundary of  $H$ . If  $M$  is a set and  $H$  is a subset of  $M$ , then  $H$  is said to be accessible from the complement of  $M$  if there is a point  $x$  in  $H$  and an arc  $\alpha$  with  $x$  as one of its endpoints so that  $\alpha - \{x\} \subset E^2 - M$ , the set  $H$  is said to be accessible from the complement of  $M$  at the point  $x$ .

If  $H$  and  $K$  are homeomorphic plane continua, then  $H$  and  $K$  are said to be *equivalently embedded* in the plane if it is true that there is a homeomorphism  $h$  of  $E^2$  onto itself that maps  $H$  onto  $K$ .

If  $D$  is a disc and  $K$  is a Cantor set lying in  $\text{Bd}(D)$  then  $\{S_i\}_{i=1}^{\infty}$  is said to be a *defining sequence of segments* for  $K$  in  $\text{Bd}(D)$  if it is true that for each positive integer  $i$ ,  $S_i$  is an arc minus its endpoints which lies in  $\text{Bd}(D)$ , no two elements of  $\{S_i\}_{i=1}^{\infty}$  intersect, and  $K = \text{Bd}(D) - \bigcup_{i=1}^{\infty} S_i$ .

## II. Plane Indecomposable Continua with Property IM

*Theorem 2.1.* Suppose  $S$  is the square disc  $[0,1] \times [0,1]$ ,  $I$  is  $[0,1] \times \{0\}$ ,  $K$  is the Cantor set lying in  $I$  and  $D_1, D_2, \dots$  is a sequence of discs lying in  $S$  so that:

$$1) D_{n+1} \subset D_n \text{ and } D_1 = S$$

2)  $K \subset \text{Bd}(D_n)$  for all positive integers  $n$ , and

3) if  $\epsilon > 0$  and  $P$  is a point of  $I - K$  then there exists an integer  $N$  and an integer  $n > N$  so that  $P \in D_N - D_n$  and every point of  $D_n$  lies within  $\epsilon$  of the component of  $D_N - D_n$  containing  $P$ .

Then the common part  $M$  of the discs  $D_1, D_2, \dots$  is an indecomposable plane continuum uncountably many composants of which are accessible from the complement of  $M$ , in particular no two points of  $K - \{(0,0), (1,0)\}$  lie in the same component of  $M$ .

*Theorem 2.2.* Suppose that there exists sequences  $D_1, D_2, D_3, \dots$ ;  $K_1, K_2, K_3, \dots$ ;  $\{S_i^1\}_{i=1}^\infty, \{S_i^2\}_{i=1}^\infty, \{S_i^3\}_{i=1}^\infty, \dots$ ;  $R_1, R_2, R_3, \dots$ ;  $J_1, J_2, J_3, \dots$ ; and  $U_1, U_2, U_3, \dots$  so that for each  $n \in \mathbb{Z}^+$ :

1)  $D_n$  is a disc and  $D_n \subset D_{n-1}$ ;

2)  $K_n$  is a Cantor set in  $\text{Bd}(D_n)$  with  $K_n \subset K_{n+1}$  and  $\{S_i^n\}_{i=1}^\infty$  is a defining sequence of segments in  $\text{Bd}(D_n)$  for  $K_n$  which are ordered in non-increasing order by diameter,  $\text{diam}(S_i^n) < 1/n$  for all positive integers  $i$  and

$\text{Max} \{\text{diam}(S_i^n)\}_{i=1}^\infty < \text{Max} \{\text{diam}(S_i^{n-1})\}_{i=1}^\infty$ ;

3)  $R_1 = S_1^1$  and  $R_n$  is an element  $S_{i_n}^j$  of  $\bigcup_{j=1}^n (\{S_i^j\}_{i=1}^\infty)$  -  $\{R_i\}_{i=1}^{n-1}$  which has diameter  $\text{Max} \{\text{diam}(S_i^j) \mid S_i^j \neq R_k \text{ for positive integers } i, j, \text{ and } k \text{ with } j \leq n \text{ and } 0 < k < n\}$ .

4)  $J_n$  is a point lying in  $R_n, J_{n-1} \notin D_n$ , and  $U_{n-1}$  is the component of  $D_{k_n} - D_n$  containing  $J_{n-1}$  where  $k_n$  is the last integer so that  $J_{n-1} \in \text{Bd}(D_{k_n})$ ; and

5) every point of  $D_n$  lies within  $1/n$  of  $U_{n-1}$ .

Then if  $M = \bigcap_{n=1}^\infty D_n$  then  $M$  is an indecomposable plane continuum no component of which is accessible at more than one point

from the complement of  $M$ .

*Theorem 2.3.* *There exists a pseudo-arc  $M$  in the plane such that no component of  $M$  is accessible from the complement of  $M$  at more than one point.*

*Observation.* Any continuum with property IM must be indecomposable. There exist different indecomposable continua with property IM. To see this last statement, let  $D_1 = [0,1] \times [0,1]$  and let  $\alpha$  be a nonseparating plane continuum lying in  $D_1$  and which intersects  $\text{Bd}(D_1)$  in exactly one point. Then a sequence  $D_1, D_2, \dots$  can be constructed satisfying the hypothesis of theorem 2.2 so that  $\alpha \subset D_i$  for all positive integers  $i$ . Thus  $\alpha \subset M = \bigcap_{i=1}^{\infty} D_i$ . It follows then that continua with property IM need not be hereditarily indecomposable.

### III. Embeddings of the Pseudo Arc in the Plane

*Theorem 3.1.* *If  $n$  is a positive integer then there exists a pseudo arc  $M_n$  in the plane so that exactly  $n$  components of  $M_n$  are accessible at more than one point from the complement of  $M_n$ .*

*Lemma.* *Suppose  $n_1, n_2, \dots$  is a sequence of non-negative integers. There exists a pseudo-arc  $M$  in the plane and a sequence  $M_1, M_2, \dots$  of pseudo-arcs lying in  $M$  such that:*

1)  $M_i$  is a pseudo-arc in the plane and exactly  $n_i$  components of  $M_i$  are accessible from the complement of  $M_i$  at more than one point;

2) if  $i \neq j$  then  $M_i$  and  $M_j$  lie in different components  $c_i$  and  $c_j$  respectively of  $M$ ;

3) if  $c$  is a component of  $M$  and  $c$  is accessible from the

complement of  $M$  at more than one point then  $c = c_i$  for some positive integer  $i$ ;

4) if  $z_i$  is the set of points of  $c_i$  at which  $c_i$  is accessible from the complement of  $M$  then  $\overline{z_i} = M_i$ ; and

5) if  $H$  is a component of  $M_i$  which is accessible from the complement of  $M_i$  at more than one point then the point  $p$  of  $H$  is accessible from the complement of  $M$  if and only if  $p$  is accessible from the complement of  $M_i$ .

*Theorem 3.2.* There are  $c$  inequivalent embeddings of the pseudo-arc in the plane, where  $c$  is the cardinality of the real numbers.

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