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1. Introduction

A *continuum* is a compact connected metric space. The letter X will always denote a continuum with metric d, and C(X) is the hyperspace of nonempty subcontinua of X metrized by the Hausdorff metric H. For basic facts about hyperspaces, see [12]. If $A \in C(X)$, then $C(A) = \{Y \in C(X) | Y \subseteq A\}$ and $\hat{A} = \{\{a\} | a \in A\}$. A continuous map $\mu: C(X) \to R$ is called a *Whitney map* if it satisfies: (1) $\mu(\{x\}) = 0$ for each $x \in X$, and (2) if $A \subseteq B$ and $A \neq B$, then $\mu(a) < \mu(B)$. Whitney [16] has shown that such maps always exist. Throughout this paper, μ will stand for an arbitrary Whitney map on C(X). It is known [4] that μ is monotone, i.e., $\mu^{-1}(t)$ is a subcontinuum of C(X) for each t. The continua $\mu^{-1}(t)$ are called the *Whitney continua*. Notice that if $A \in C(X)$, then $C(A) \cap \mu^{-1}(t)$ is a continuum since it is a Whitney continuum in C(A).

A topological property P is said to be a *Whitney property* provided that whenever a continuum X has property P, so does $\mu^{-1}(t)$ for each Whitney map μ for C(X) and each t with $0 < t < \mu(X)$. Whitney properties were investigated by several authors (see [8], [14], [15], and, for a summary of results, see [12]). Nadler [12] defines a topological

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property P to be a *strong Whitney-reversible* property (resp., *Whitney-reversible property*) provided that whenever X is a continuum such that $\mu^{-1}(t)$ has property P for some Whitney map (resp., all Whitney maps) μ for C(X), and all t with $0 < t < \mu(X)$, then X has property P. Nadler ([12], [13]) has shown that some topological properties are Whitney-reversible and he asked [12, (14.57)] if certain other properties are Whitney-reversible. In section 2 we show that hereditary decomposability, hereditary arcwise connectedness, and C*-smoothness are strong Whitney-reversible properties.

In section 3 we study the relation between convexity of the Whitney continua and that of the underlying continuum.

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2. Whitney-Reversible Properties

A continuum is said to be *decomposable* provided that it is the union of two proper subcontinua. It is said to be *indecomposable* provided that it is not decomposable. A property P of a continuum X is said to be *hereditary* provided that each subcontinuum of X has P. We will denote by σ the union function $\sigma: C(C(X)) \to C(X)$ defined by $\sigma(\alpha) = \cup\{A | A \in \alpha\}$, and by $\hat{\imath}$ the function $\hat{\imath}: C(X) \to C(C(X))$ defined by $\hat{\imath}(A) = \hat{A}$. It is known that σ is continuous [6], and that $\hat{\imath}$ is an isometry [12, (16.6)].

It is known [12, p. 413] that indecomposability is not a Whitney property. However, this result shows that indecomposability of X is reflected in $\mu^{-1}(t)$.

2.1. *Theorem. Let* X *be an indecomposable continuum. Let* μ *be a Whitney map for* C(X). *Then for each* $t \in (0,\mu(X))$ *there exists an indecomposable continuum* $\beta_t \subseteq \mu^{-1}(t)$ *such that* $\sigma(\beta_t) = X$.

Proof. Let $t \in (0,\mu(X))$ be fixed. It follows by the continuity of the union function σ and Brouwer's reduction theorem that $\mu^{-1}(t)$ contains a continuum β_t which is irreducible with respect to the property that $\sigma(\beta_t) = X$. We show that β_t is indecomposable. For, if β_t were the union of two proper subcontinua β_1 and β_2, then $\sigma(\beta_1)$ and $\sigma(\beta_2)$ would be proper subcontinua of X such that $X = \sigma(\beta_1) \cup \sigma(\beta_2)$. This contradicts the fact that X is indecomposable.

It is known (see [12, p. 454]) that decomposability is not a Whitney property.

2.2. *Theorem. Assume there is a sequence* $\{t_n\}_{n\in\omega}$ *such that* $t_n \to 0$ *as* $n \to \infty$, *and* $\mu^{-1}(t_n)$ *is hereditarily decomposable for each* $n = 1,2,3,\cdots$, *then* X *is hereditarily decomposable. Hence, hereditary decomposability is a strong Whitney-reversible property.*

Proof. Suppose on the contrary that X contains an indecomposable continuum Y. It follows easily from the continuity of μ, and the hypothesis of the theorem that there exists $t_o \in \{t_n | n \in \omega\}$ such that $C(Y) \cap \mu^{-1}(t_o)$ is a nondegenerate subcontinuum of $\mu^{-1}(t_o)$. Then, by 2.1, there exists an indecomposable continuum $\beta \subseteq C(Y) \cap \mu^{-1}(t_o)$. This contradicts the fact that $\mu^{-1}(t_o)$ is hereditarily decomposable.

The result just proved answers one of the questions in [12, (14.57)].

A continuum X is *unicoherent* provided that $A \cap B$ is connected whenever A and B are subcontinua of X such that $A \cup B = X$. A *triod* is a continuum M which contains a subcontinuum N such that the complement of N in M is the union of three nonempty mutually separated sets. A continuum is *a-triodic* provided it contains no triod. A continuum X is *chainable* provided that for each $\varepsilon > 0$, there exists a continuous function $f: X \to R$ such that $\operatorname{diam}(f^{-1}(r)) < \varepsilon$ for each $r \in f(X)$.

Nadler has proved the following result (see [12, (14.46), (14.49-51)].

2.3. *Theorem* [Nadler]. *Assume there is a sequence* $\{t_n\}_{n\in\omega}$ *such that* $t_n \to 0$ *as* $n \to \infty$ *and* $\mu^{-1}(t_n)$ *is unicoherent (or, respectively, a-triodic, an arc, a circle), then* X *is unicoherent (or, respectively, a-triodic, an arc, a circle).*

The following two results provide partial answers to the question of whether chainability is a Whitney-reversible property.

2.4. *Theorem. Assume there is a sequence* $\{t_n\}_{n\in\omega}$ *such that* $t_n \to 0$ *as* $n \to \infty$ *and* $\mu^{-1}(t_n)$ *is an hereditarily decomposable chainable continuum for each* $n = 1,2,3,\cdots$, *then* X *is an hereditarily decomposable chainable continuum.*

Proof. It follows by 2.2 that X is hereditarily decomposable. Since a chainable continuum is hereditarily unicoherent and a-triodic, it follows by 2.3 that X is hereditarily unicoherent and a-triodic. Bing [2, Theorem 11] has proved that an hereditarily decomposable continuum is chainable if and only if it is a-triodic and hereditarily unicoherent.

A continuum X is said to have *property*[κ] provided that for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $a,b \in X$, $d(a,b) < \delta$, and $a \in A \in C(X)$, then there exists $B \in C(X)$ such that $b \in B$, and $H(A,B) < \varepsilon$. It is known [6] that if X has property[κ], then the function $F_\mu: X \times [0,\mu(X)] \to C(C(X))$ defined by $F_\mu(x,t) = \{A \in \mu^{-1}(t) | x \in A\}$ is continuous.

2.5. *Theorem. Let* X *be a continuum which has property*[κ]. *Assume there is a sequence* $\{t_n\}_{n\in\omega}$ *such that* $t_n \to 0$ *as* $n \to \infty$, *and* $\mu^{-1}(t_n)$ *is chainable for each* $n = 1,2,\cdots$, *then* X *is chainable.*

Proof. Let $\varepsilon > 0$ be given. By the continuity of μ, and the hypothesis of the theorem, there exists $t_o \in \{t_n | n \in \omega\}$ such that $\text{diam}(M) < \varepsilon/2$ for each $M \in \mu^{-1}(t_o)$. Since $\mu^{-1}(t_o)$ is chainable, there exists a continuous map $g: \mu^{-1}(t_o) \xrightarrow{\text{onto}} [0,1]$ such that $\text{diam}(g^{-1}(r)) < \varepsilon/2$ for each $r \in [0,1]$. Define $f: X \to [0,1]$ by $f(x) = \text{centre}(g(F_\mu(x,t_o)))$. Since X has property[$\kappa$], f is continuous. Let $r \in f(X)$, and let $a,b \in f^{-1}(r)$. Then there exist $A \in F_\mu(a,t_o)$ and $B \in F_\mu(b,t_o)$ such that $r = g(A) = g(B)$. Since g is an $\varepsilon/2$-map, $H(A,B) < \varepsilon/2$. Thus, $d(a,b) < \varepsilon$. This shows that f is an ε-map. Hence, X is chainable.

It is known (see [12, (14.48)]) that arcwise connectedness is not a Whitney-reversible property. Let us note the following:

2.6. *Theorem. Assume that* $\mu^{-1}(t)$ *is hereditarily arcwise connected for each* $t \in (0,\mu(X))$, *then* X *is an arc or a circle. Hence, hereditary arcwise connectedness is a strong Whitney-reversible property.*

Proof. It is known [9, p. 212] that each arcwise connected continuum is decomposable. Thus, each $\mu^{-1}(t)$ is hereditarily decomposable for each $t \in (0,\mu(X))$. Then, by 2.2, X is hereditarily decomposable. It follows by [8, (3.3)] that X is a-triodic. Now, we show that $C(X)\setminus\{E\}$ is arcwise connected for each proper subcontinuum E of X. Let E be an arbitrary but fixed subcontinuum of X. We may assume that E is non-degenerate. To prove that $C(X)\setminus\{E\}$ is arcwise connected, it suffices from the arc structure of C(X) to show that if A is a proper subcontinuum of E, then A and X can be joined by an arc in $C(X)\setminus\{E\}$. Let $t > 0$ be chosen such that $\mu(A) \leq t < \mu(E)$. Let $B \in \mu^{-1}(t)$ such that $A \subseteq B$, and let α_1 be an order arc from A to B (see [12]). Let $g \in X\setminus E$, and let $G \in \mu^{-1}(t)$ such that $g \in G$. Since $\mu^{-1}(t)$ is arcwise connected, there exists an arc α_2 joining B and G in $\mu^{-1}(t)$. Let α_3 be an order arc from G to X. It follows that $\alpha_1 \cup \alpha_2 \cup \alpha_3$ is an arc joining A and X in $C(X)\setminus\{E\}$. This shows that $C(X)\setminus\{E\}$ is arcwise connected. Since X is a-triodic and hereditarily decomposable, it follows by [12, (11.16)] that X is chainable or circle-like.

If X is chainable, then since the property of being a chainable continuum is a Whitney property [7], each $\mu^{-1}(t)$ is chainable, $0 < t < \mu(X)$. Since each arcwise connected chainable continuum is an arc, each $\mu^{-1}(t)$ is an arc. Then, by 2.3, X is an arc. On the other hand, if X is circle-like and not chainable (i.e., proper circle-like), then since the property of being a proper circle-like continuum is a Whitney property [7], each $\mu^{-1}(t)$ is a proper circle-like continuum, $0 < t < \mu(X)$. Thus, each $\mu^{-1}(t)$ is an hereditarily arcwise connected circle-like continuum. By [11, Theorem 6], each $\mu^{-1}(t)$ is a circle. Thus, by 2.3, X is a circle.

A continuum X is said to be C*-*smooth* provided that the function C*: $C(X) \to C(C(X))$ defined by $C^*(A) = C(A)$ is continuous [12, (15.5)].

We denote by H^2 the Hausdorff metric on C(C(X)) corresponding to H as a metric on C(X), and by H^3 the Hausdorff metric on C(C(C(X))) corresponding to H^2 as a metric on C(C(X)).

2.7. *Lemma. For each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $t < \delta$, A *is any subcontinuum of* X, $\mu^{-1}(t)$ *is hereditarily unicoherent, and* β *is any subcontinuum of* $\mu^{-1}(t)$ *such that* $\sigma(\beta) = A$, *then* $H^3(C(\hat{A}),C(\beta)) \leq \varepsilon$.

Proof. Let $\varepsilon > 0$ be given. By the continuity of μ and the compactness of C(X), there exists $\delta > 0$ such that if $0 < t < \delta$, and $M \in \mu^{-1}(t)$, then $\operatorname{diam}(M) < \varepsilon$. Assume that $\mu^{-1}(t)$ is hereditarily unicoherent for some $t < \delta$. Let $A \in C(X)$, and let $\beta \subset C(\mu^{-1}(t))$ such that $\sigma(\beta) = A$. Now,

$$H^3(C(\hat{A}),C(\beta)) = \max\{\sup_{M\in C(\beta)}(\inf_{N\in C(\hat{A})} H^2(M,N)),\ \sup_{N\in C(\hat{A})}(\inf_{M\in C(\beta)} H^2(M,N))\}.$$

If $M \in C(\beta)$, let $N = (\hat{\sigma}M)$. Then it is easy to see that $H^2(M,N) < \varepsilon$. On the other hand, if $N \in C(\hat{A})$, let $X(\sigma(N),\mu,t) = \{G \in \mu^{-1}(t) | G \cap \sigma(N) \neq \emptyset\}$. Then, by [8, (3.2)], $X(\sigma(N),\mu,t)$ is a subcontinuum of $\mu^{-1}(t)$. Let

$M = X(\sigma(N), \mu, t) \cap \beta$. Since $\mu^{-1}(t)$ is hereditarily unicoherent, M is a continuum, and once again $H^2(M,N) < \varepsilon$. This shows that $H^3(C(\hat{A}), C(\beta)) \leq \varepsilon$.

2.8. *Example.* The following example shows that the assumption that $\mu^{-1}(t)$ is hereditarily unicoherent cannot be dropped from Lemma 2.7. Let X be the unit circle, and let μ be any Whitney map for $C(X)$. Note that $\mu^{-1}(t)$ is a circle for each $t \in (0, \mu(X))$ [7]. Let $\varepsilon = 1/10$. We show that for any $t \in (0, \mu(X))$, there exists a subcontinuum $\beta \subseteq \mu^{-1}(t)$ such that $\sigma(\beta) = X$, and $H^3(C(\hat{X}), C(\beta)) > 1/10$. Let $t \in (0, \mu(X))$ be arbitrary but fixed. It suffices to assume that $\mathrm{diam}(M) < 1/4$ for each $M \in \mu^{-1}(t)$. Let $\ell > 0$ such that $\mathrm{diam}(M) > \ell$ for each $M \in \mu^{-1}(t)$. Let S be an open interval of X of length ℓ, and let $X_1 = X \setminus S$. Let $\beta = \{M \in \mu^{-1}(t) \mid M \cap X_1 \neq \emptyset\}$. Then β is a subcontinuum of $\mu^{-1}(t)$ such that $\sigma(\beta) = X$. Let N be the arc of X of length $= 1$ which contains S in its middle. It is easy to see that $H^2(N, \gamma) > 1/10$ for each subcontinuum $\gamma \subseteq \beta$, and consequently $H^3(C(\hat{X}), C(\beta)) > 1/10$.

2.9. *Theorem. Assume there is a sequence* $\{t_n\}_{n\in\omega}$ *such that* $t_n \to 0$ *as* $n \to \infty$, *and* $\mu^{-1}(t_n)$ *is* C*-*smooth for each* $n = 1, 2, \cdots$. *Then,* X *is* C*-*smooth. Hence,* C*-*smoothness is a strong Whitney-reversible property.*

Proof. Let $\{A_n\}_{n\in\omega}$ be a sequence in $C(X)$ such that $\lim_{n\to\infty} A_n = A$. To prove that X is C*-smooth, it suffices to show that if $\{C(A_{n_j})\}_{j\in\omega}$ is any convergent subsequence of the sequence $\{C(A_n)\}_{n\in\omega}$, then $\lim_{j\to\infty} C(A_{n_j}) = C(A)$. We may assume that A is non-degenerate. Let $\Lambda = \lim_{j\to\infty} C(A_{n_j})$, and let $\varepsilon > 0$

be arbitrary. Let $\delta > 0$ be chosen as in Lemma 2.7 with ε replaced by $\varepsilon/3$. Let $t \in \{t_n | n \in \omega\}$ such that $t < \delta$, and such that $C(A) \cap \mu^{-1}(t)$ is a non-degenerate continuum. Then, by [5, (2.1)], $\lim_{j\to\infty}(C(A_{n_j}) \cap \mu^{-1}(t)) = \Lambda \cap \mu^{-1}(t)$. Since $\mu^{-1}(t)$ is C*-smooth, there exists a natural number N such that for each $j \geq N$,

$$H^3(C(C(A_{n_j}) \cap \mu^{-1}(t)), C(\Lambda \cap \mu^{-1}(t))) < \varepsilon/3. \tag{1}$$

We may assume that for each $j \geq N$, $\sigma(C(A_{n_j}) \cap \mu^{-1}(t)) = A_{n_j}$. Since each C*-smooth continuum is hereditarily unicoherent [5], it follows by 2.7 that

$$H^3(C(A_{n_j}) \cap \mu^{-1}(t)), C(\hat{A}_{n_j})) \leq \varepsilon/3. \tag{2}$$

Since the union function σ is continuous, $A = \sigma(\Lambda \cap \mu^{-1}(t))$. Hence, by 2.7

$$H^3(C(\hat{A}), C(\Lambda \cap \mu^{-1}(t)) \leq \varepsilon/3. \tag{3}$$

It follows from (1), (2), and (3) and the triangle inequality that $H^3(C(\hat{A}), C(\hat{A}_{n_j})) < \varepsilon$ for each $j \geq N$. Since for each $M \in C(\hat{A})$, and each $N \in C(\hat{A}_{n_j})$, $H^2(M,N) = H(\sigma(M),\sigma(N))$, it follows that $H^2(C(A), C(A_{n_j})) < \varepsilon$ for each $j \geq N$. Consequently, $H^2(C(A),\Lambda) < \varepsilon$. Since ε is arbitrary, $\Lambda = C(A)$ and the proof is complete.

2.10. *Remark*. In contrast with 2.9, let us show that C*-smoothness is not a Whitney property. By [12, (15.11)] a locally connected continuum is C*-smooth if and only if it is a dendrite. Let X be a simple triod (a continuum homeomorphic to $\{(0,y) \in R^2 | 0 \leq y \leq 1\} \cup \{(x,1) \in R^2 | -1 \leq x \leq 1\}$). Then X is C*-smooth. It follows by [12, (14.9)] that $\mu^{-1}(t)$ is a locally connected continuum for each $t \in (0,\mu(X))$. It is easy to see that $\mu^{-1}(t)$ contains a 2-cell for each $t \in (0,\mu(X))$, and, therefore, $\mu^{-1}(t)$ is not C*-smooth.

3. Convexity

A continuum X is said to be convex provided that for each pair of points $x,y \in X$, there exists a point $z \in X\setminus\{x,y\}$ such that $d(x,z) + d(z,y) = d(x,y)$. It is known that if X is convex, then each pair of points of X can be joined by a segment in X.

Let us note the following theorem for which we will show the converse is false.

3.1. *Theorem. Assume there is a sequence* $\{t_n\}_{n\in\omega}$ *such that* $t_n \to 0$ *as* $n \to \infty$, *and* $\mu^{-1}(t_n)$ *is convex (with respect to the Hausdorff metric), then* X *is convex (with respect to the original metric* d *on* X*).*

Proof. Since μ is an open map [4], and $\lim_{n\to\infty} t_n = 0$, $\lim_{n\to\infty} \mu^{-1}(t_n) = \hat{X}$. Since each $\mu^{-1}(t_n)$ is convex, it follows by [3, (4.8)] that $\hat{X}$ is convex, and consequently X is convex.

3.2. *Example*. The following is an example of a convex arc X, and a Whitney map μ for C(X), such that $\mu^{-1}(t)$ is not convex for any $t \in (0,1]$. Let $X = [0,3]$ with the Euclidean metric. Define a homeomorphism $f\colon [0,3] \to [0,6]$ as follows:

$$f(x) = \begin{cases} x\ , & \text{if } x \in [0,1] \\ x^2, & \text{if } x \in [1,2] \\ 2x, & \text{if } x \in [2,3]. \end{cases}$$

Define $\mu: C(X) \to [0,\infty)$ by $\mu([a,b]) = f(b) - f(a)$. Then, μ is a Whitney map for $C(X)$. We show that $\mu^{-1}(t)$ is not convex. Let $t \in (0,1]$ be fixed. Let $A = [0,t]$, $B = [3-t/2,3]$, and $D = [1,\sqrt{1+t}]$. Then A, B and $D \in \mu^{-1}(t)$. It is known that $\mu^{-1}(t)$ is an arc [7]. Note that A and B are the end points of $\mu^{-1}(t)$. It is easy to see that $H(A,D) = 1$, $H(D,B) = 3 - \sqrt{1+t}$, and $H(A,B) = 3 - t/2$. Thus, $H(A,B) \neq H(A,D) + H(D,B)$. This shows that $\mu^{-1}(t)$ is not convex.

3.3. *Remark.* It is known [1] that a convex continuum is locally connected, and that local connectedness is a Whitney property [12, (14.9)]. Bing [1] and Moise [10] have shown independently that every locally connected continuum admits a convex metric. In view of these facts, we see that if X is a convex continuum, $\mu^{-1}(t)$ admits a convex metric. However, as 3.2 shows, it may happen that $\mu^{-1}(t)$ is not convex with respect to the Hausdorff metric.

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