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1. Introduction

In [19] D. M. Hyman introduced the class of ANR-divisors i.e. continua X such that Y/X is an ANR for each ANR-space Y containing X. By using Chapman's Complement Theorem [10] it is easy to show that being an ANR-divisor is a shape invariant (see [9]). More generally we have: being an ANR-divisor is a hereditary strong shape invariant in the sense of Edwards-Hastings [14] (see [9]). By using the work of Hyman [19] and characterizations of pointed FANR's it is clear that pointed FANR's are ANR-divisors. However an example in [12] shows that the class of ANR-divisors is wider than the class of pointed FANR's.

In this paper we introduce the class of LC^n-divisors in analogy to ANR-divisors. We give a characterization of LC^n-divisors which implies that being an LC^n-divisor is a hereditary shape invariant. As a consequence we infer that being an ANR-divisor is a hereditary shape invariant in the class of continua of finite fundamental dimension.

We assume that the reader is familiar with some elementary facts from shape theory (see [6], [22], [23] and [27]) and from the theory of pro-categories (see [2], [11] and [14]).

2. Some Algebraic Preliminaries

For a definition and basic properties of pro-categories (see [2], [11] and [14]).

Recall that an inverse sequence $\underline{A} = (A_n, p_n^{n+1})$ of groups is said to satisfy the Mittag-Leffler condition provided for each n there exists k > n such that

$$\text{im } p_n^k = \text{im } p_n^m \quad \text{for} \quad m > k$$

(see [3] and [26]).

$\underline{A}$ is said to be stable iff $\underline{A}$ is isomorphic to a group in the category of pro-groups pro-Gr (see [11] and [14]).

For a definition of $\varprojlim^1 \underline{A}$ and its properties see [8], pp. 250-252.

Lemma 2.1. *Let* $\underline{A} = (A_n, p_n^{n+1})$ *and* $\underline{B} = (B_n, q_n^{n+1})$ *be inverse sequences of groups and let* $f_n\colon A_n \to B_n$ *be homomorphisms such that* $q_n^{n+1} f_{n+1} = f_n p_n^{n+1}$ *for* $n \geq 1$.

If $\varprojlim^1 \underline{A} = *$, *then* $\varprojlim^1 \underline{C} = *$, *where* $\underline{C} = (\text{im } f_n, r_n^{n+1})$ *and* r_n^{n+1} *is induced by* q_n^{n+1}.

If $\underline{A}$ *and* $\underline{B}$ *are stable, then* $\underline{D} = (\ker f_n, s_n^{n+1})$ *is stable, where* s_n^{n+1} *is induced by* p_n^{n+1}.

Proof. Suppose $\varprojlim^1 \underline{A} = *$. Since

$$0 \to \ker f_n \to A_n \to \text{im } f_n \to 0$$

is exact for each n, then the following sequence is exact:

$$0 \to \varprojlim \underline{D} \to \varprojlim \underline{A} \to \varprojlim \underline{C} \to \varprojlim^1 \underline{D} \to \varprojlim^1 \underline{A} \to \varprojlim^1 \underline{C} \to 0$$

(see [8], p. 252).

Hence, $\varprojlim^1 \underline{C} = *$.

Suppose $\underline{A}$ and $\underline{B}$ are stable. Then we may assume that

$$p_n^{n+1}/\text{im } p_{n+1}^{n+2}\colon \text{im } p_{n+1}^{n+2} \to \text{im } p_n^{n+1} \quad \text{and}$$

$$q_n^{n+1}/\text{im } q_{n+1}^{n+2}\colon \text{im } q_{n+1}^{n+2} \to \text{im } q_n^{n+1}$$

are isomorphisms for each n.

Let $x \in \ker f_{n+1}$. Then there is $y \in A_{n+3}$ with $p_n^{n+3}(y) = p_n^{n+1}(x)$. Hence, $q_n^{n+2}q_{n+2}^{n+3}f_{n+3}(y) = q_n^{n+3}f_{n+3}(y) = f_n p_n^{n+3}(y) = f_n p_n^{n+1}(y) = q_n^{n+1}f_{n+1}(x) = 0$. Therefore, $f_{n+2}p_{n+2}^{n+3}(y) = q_{n+2}^{n+3}f_{n+3}(y) = 0$ i.e. $p_{n+2}^{n+3}(y) \in \ker f_{n+2}$. Since $p_n^{n+2}p_{n+2}^{n+3}(y) = p_n^{n+3}(y) = p_n^{n+1}(x)$ we get $p_n^{n+2}(\ker f_{n+2}) = p_n^{n+1}(\ker f_{n+1})$. Hence

$$s_n^{n+1}/\operatorname{im} s_{n+1}^{n+2}: \operatorname{im} s_{n+1}^{n+2} \to \operatorname{im} s_n^{n+1}$$

is an isomorphism for each n and $\underline{D}$ is stable.

Let $p_n^{k-1,k}: G_n^k \to G_n^{k-1}$ and $q_{n,n+1}^k: G_{n+1}^k \to G_n^k$ be homomorphisms of groups ($n \geq 1$ and k-integer) such that

$$p_n^{k-1,k}q_{n,n+1}^k = q_{n,n+1}^{k-1}p_n^{k-1,k}.$$

Suppose that each sequence $\underline{G}_n = (G_n^k, p_n^{k-1,k})$ is exact and let $\underline{G}^k = (G_n^k, q_{n,n+1}^k)$ for each k.

Lemma 2.2. *If* $\underleftarrow{\lim}^1 \underline{G}^k = *$ *for each* k, *then the sequence*

$$\dots \to \lim \underline{G}^k \to \lim \underline{G}^{k-1} \to \lim \underline{G}^{k-2} \to \dots$$

is exact.

Proof. Analogous to the corresponding result in [30], where Lemma 2.2 is proved in case where $\underline{G}^k$ satisfy the Mittag-Leffler condition.

Lemma 2.3. *If* $\underline{G}^i$ *are stable for* $i = 0,1$ *and* $\underleftarrow{\lim}^1 \underline{G}^3 = *$, *then* $\underleftarrow{\lim}^1 \underline{G}^2 = *$.

Proof. For each n we have the following exact sequence

$$0 \to \operatorname{im} p_n^{2,3} \to G_n^2 \to \ker p_n^{0,1} \to 0.$$

By Proposition 2.3 in [8] (p. 252) there is the following exact sequence $0 \to \varprojlim(\mathrm{im}\, p_n^{2,3}) \to \varprojlim \underline{G}^2 \to \varprojlim(\ker p_n^{0,1}) \to \varprojlim^1(\mathrm{im}\, p_n^{2,3}) \to \varprojlim^1 \underline{G}^2 \to \varprojlim^1(\ker p_n^{0,1}) \to 0$. By Lemma 2.1 the inverse sequence $(\ker p_n^{0,1})$ is stable and $\varprojlim^1(\mathrm{im}\, p_n^{2,3}) = *$. Hence $\varprojlim^1(\ker p_n^{0,1}) = *$ (see [8], p. 252) and consequently $\varprojlim^1 \underline{G}^2 = *$.

Lemma 2.4. *If* $\underline{G}^i$ *is isomorphic to an inverse sequence of countable groups for* $i = 1,3$, *then* $\underline{G}^2$ *is isomorphic in* pro-Gr *to an inverse sequence of countable groups.*

Proof. An inverse sequence (A_n, r_n^m) of groups is isomorphic to an inverse sequence of countable groups iff for each n there is $m > n$ such that $r_n^m(A_m)$ is countable.

Let $n \geq 1$ and take $k > m > n$ such that $q_{n,m}^i(G_m^i)$ is countable for $i = 1,3$ and $q_{m,k}^i(G_k^i)$ is countable for $i = 1,3$. Take elements $a_i \in G_m^2$, $i \geq 1$, such that each element in $p_m^{1,2}(G_m^2) \cap q_{m,k}^1(G_k^1)$ is equal to $p_m^{1,2}(a_i)$ for some i.

Let $a \in G_k^2$ be an arbitrary element. Then $q_{m,k}^1 p_k^{1,2}(a) = p_m^{1,2}(a_i)$ for some i. Thus $p_m^{1,2}(a_i q_{m,k}^2(a^{-1})) = 0$ and there is $b \in G_m^3$ with $p_m^{2,3}(b) = a_i q_{m,k}^2(a^{-1})$. Then $q_{n,k}^2(a) = q_{n,k}^2 p_m^{2,3}(b^{-1}) q_{n,m}^2(a_i) = p_n^{2,3} q_{n,m}^3(b^{-1}) q_{n,m}^2(a_i)$. This implies that $q_{n,k}^2(G_k^2)$ is countable because $q_{n,m}^3(G_m^3)$ is countable.

Lemma 2.5. *If* $\underline{G} = (G_n, r_n^{n+1})$ *is isomorphic to an inverse sequence of countable groups and* $\varprojlim^1 \underline{G} = *$, *then* $\underline{G}$ *satisfies* the Mittag-Leffler condition.

Proof. Take an increasing sequence $(n_k)_{k=1}^\infty$ of natural numbers such that

$$r_{n_k}^{n_{k+1}}(G_{n_{k+1}})$$

is countable for each k and $n_1 = 1$.

Define $\underline{H} = (H_n, s_n^{n+1})$ as follows: $H_n = 0$ for $1 \leq n \leq n_2$ and $H_n = \operatorname{im} r_{n_{k-1}}^n$ for $n_k < n \leq n_{k+1}$ and $k \geq 2$, s_n^{n+1}: $H_{n+1} \to H_n$ is the inclusion homomorphism if $n_k < m < m+1 \leq n_{k+1}$ for some k and s_n^{n+1} is induced by $r_{n_{k-2}}^{n_{k-1}}$ if $n = n_k$ for some k. Let f_n: $G_n \to H_n$ be induced by $r_{n_{k-1}}^n$ if $n_k < n \leq n_{k+1}$ and $k \geq 2$ or be the zero homomorphism if $n \leq n_2$.

Then each H_n is countable and $H_n = f_n(G_n)$. By Lemma 2.1 $\underleftarrow{\lim}^1 \underline{H} = *$ and R. Geoghegan [15] has proved that $\underline{H}$ satisfies the Mittag-Leffler condition in such a case (see [16] for the Abelian case). Since $\underline{G}$ and $\underline{H}$ are isomorphic, then $\underline{G}$ satisfies the Mittag-Leffler condition as well.

3. Properties of LC^n-Spaces

From now on by $H_k(X)$ we denote the reduced singular homology group of a space X and by $\check{H}_k(X)$ we denote the reduced Čech homology group of X (all groups are taken with integer coefficients).

In the sequel we shall need the following:

Theorem 3.1. (see Theorem V.2.1 and Propositions II.9.1, II.10.1 in [17]). *Let* X *and* Y *be metrizable spaces such that* $X \cap Y$ *is a closed subset of both* X *and* Y. *If* X, Y *and* $X \cap Y$ *are* LC^n*-spaces, then* $X \cup Y$ *is an* LC^n*-space. If* $X \cup Y$ *and* $X \cap Y$ *are* LC^n*-spaces, then* X *and* Y *are* LC^n*-spaces.*

The following result of W. Hurewicz [18] is basic in our considerations.

Theorem 3.2. *An* LC^1*-space* X *is an* LC^n*-space* ($n \geq 2$) *iff for each* $x \in X$ *and for each neighborhood* U *of* x *in* X *there is a neighborhood* V *of* x *in* U *such that the inclusion map* $i: V \to U$ *induces trivial homomorphism*

$$\check{H}_k(i): \check{H}_k(V) \to \check{H}_k(U)$$

for $k \leq n$.

Theorem 3.3. *If* X *is a connected metrizable* LC^n*-space, then the natural morphism from* $H_k(X)$ *to pro-*$H_k(X)$ *is an isomorphism for* $k \leq n$ *and an epimorphism for* $k = n+1$.

Proof. Take $x \in X$. Theorem 8.7 in [11] says that the natural morphism from $\pi_k(X,x)$ to pro-$\pi_k(X,x)$ is an isomorphism for $k \leq n$ and an epimorphism for $k = n+1$. Hence if $f: \mathrm{Sin}(X,x) \to \check{C}(X,x)$ is the natural morphism from the geometric realization of the singular complex of (X,x) to the Čech system of (X,x), then pro-$\pi_k(f)$ is an isomorphism for $k \leq n$ and an epimorphism for $k = n+1$. Now results in [24] and [28] imply that pro-$H_k(f)$ is an isomorphism for $k \leq n$ and an epimorphism for $k = n+1$ which concludes the proof.

Lemma 3.4. *Let* A *be a closed subset of a metrizable space* X. *If* $(A_n)_{n=1}^{\infty}$ *is a decreasing sequence of subsets of* X *such that for any neighborhood* W *of* A *in* X *there is* A_n *with* $A \subset A_n \subset W$, *then*

a) pro-$H_k(A)$ *is an inverse limit of (pro-*$H_k(A_n)$, *pro-*$H_k(i_n^{n+1})$) *in pro-*Gr, *where* $i_n^{n+1}: A_{n+1} \to A_n$ *is the inclusion,*

b) $\check{H}_k(A)$ *is the inverse limit of* $(\check{H}_k(A_n), \check{H}_k(i_n^{n+1}))$, *c) if pro-*$H_k(A_n)$ *is stable for each* n, *then pro-*$H_k(A)$ *is isomorphic to* $(\check{H}_k(A_n),\check{H}_k(i_n^{n+1}))$ *in pro-*Gr.

Proof. It follows from the assumptions that A is an inverse limit of $(A_n, S(i_n^{n+1}))$ in the shape category, where S denotes the shape functor (see [21] and [22]).

A description of inverse limits in pro-categories given in [2] implies that for any functor F: C → D the corresponding functor pro-F: pro-C → pro-D is continuous i.e. preserves inverse limits. Taking $F = H_k$ we get that Condition a holds. The Condition b follows from Condition a and the fact that the inverse limit functor $\varprojlim$: pro-Gr → Gr is continuous. The Condition c is a consequence of Conditions a and b.

4. LC^n-Divisors

Definition. A continuum X is said to be an LC^n-divisor provided for each LC^n-space Y containing X the quotient space Y/X is an LC^n-space.

It is proved in [4] (see also [1]) that

Proposition 4.1. *Each* FAR-*space* X *is an* LC^n-*divisor for all* n.

We need the following to show that X is an LC^n-divisor iff Y/X is an LC^n-space for some LC^n-space Y containing X (for each space Z we denote by C(Z) a cone over Z).

Lemma 4.2. *If* $X \subset Y \subset Q$ *are subcontinua of the Hilbert cube* Q *such that* Y *and* Y/X *are* LC^n-*spaces, then* Q/X *is an* LC^n-*space.*

Proof. By Theorem 3.1 $Q \cup C(Y)$ is an LC^n-space and by Proposition 4.1 $(Q \cup C(Y))/C(X)$ is an LC^n-space. Since

$(Q \cup C(Y))/C(X) = (Q/X) \cup (C(Y)/C(X))$ and $(Q/X) \cap (C(Y)/C(X)) = Y/X$, then by Theorem 3.1 Q/X is an LC^n-space. Thus the proof of Lemma 4.2 is concluded.

If $A \subset X$ are subsets of a compact space Z, then we consider $X \cup C(A)$ as a space with topology induced from $C(Z)$. Since $C(A)$ is contractible, then the inclusion from $X \cup C(A)$ into the pair $(X \cup C(A), C(A))$ induces isomorphisms of all reduced singular homology groups. By the excision property of singular homology (see [32]) we get that the inclusion from (X,A) to $(X \cup C(A), C(A))$ induces isomorphisms of all reduced singular homology groups. From the exact sequence of homology groups for the pair (X,A) we get the following exact sequence

$$\ldots \to H_k(A) \to H_k(X) \to H_k(X \cup C(A)) \to H_{k-1}(A) \to \ldots .$$

Moreover, if $B \subset Y \subset X$ and $B \subset A$, then the diagram

$$\begin{array}{ccccccccc} \ldots \to & H_k(B) & \to & H_k(Y) & \to & H_k(Y \cup C(B)) & \to & H_{k-1}(B) & \to \ldots . \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \ldots \to & H_k(A) & \to & H_k(X) & \to & H_k(X \cup C(A)) & \to & H_{k-1}(A) & \to \ldots . \end{array}$$

is commutative.

Recall that a continuum X is nearly 1-movable ([25]) provided for each neighborhood U of X in the Hilbert space Q there is a neighborhood V of X in U such that for any loop $f: S^1 = \partial \Delta^2 \to V$ and for each neighborhood W of X in Q there is a finite disjoint collection of discs D_i in $\operatorname{Int} \Delta^2$ and an extension of f to

$$\bar{f}: (\Delta^2 - \cup \operatorname{Int} D_i, \cup \partial D_i) \to (U, U \cap W).$$

Theorem 4.3. *If* X *is a nearly* 1-*movable continuum such* *that pro-*$H_k(X)$ *is stable for* $k < m$ *and satisfies the Mittag-Leffler condition for* $k = m (m \geq 1)$, *then* Y/X *is an* LC^m*-space for each* LC^m*-space* Y *containing* X.

Proof. Let $(U_n)_{n=1}^{\infty}$ be a decreasing sequence of open neighborhoods of X in Y such that $U_{n+1} \subset cl(U_{n+1}) \subset U_n$ and $X = \cap U_n$. By Theorem 3.3 and Lemma 3.4 the inverse sequence $(H_k(U_n), H_k(i_n^{n+1}))$ is stable for $k < m$ and satisfies the Mittag-Leffler condition for $k = m$.

Fix $n \geq 1$. Then for each $r > n$ there is the following exact sequence

$$0 \to A_r \to H_m(U_n) \to \dots \to H_k(U_n) \to H_k(U_n \cup C(U_r)) \to H_{k-1}(U_r) \to \dots,$$

where $A_r = \operatorname{im} H_m(i_n^r) \subset H_m(U_n)$. Since $(H_m(U_r), H_m(i_r^s))$ satisfies the Mittag-Leffler condition, then $\varprojlim^1 A_r = *$ by Lemma 2.1. By Lemma 2.3 we get that $\varprojlim^1 (H_k(U_n \cup C(U_r), H_k(j_r^s)) = *$, where j_r^s is the inclusion map. By Lemma 2.2 the following sequence is exact

$$0 \to B_n \to H_m(U_n) \to \dots \to H_k(U_n) \to \check{H}_k(U_n \cup C(X)) \to \check{H}_{k-1}(X) \to \dots,$$

where $B_n = \varprojlim A_r$ is the image of $\check{H}_m(X)$ in $H_m(U_n)$. Observe that the diagram

$$\begin{array}{ccccccccccccc} 0 \to & B_p & \to & H_m(U_p) & \to \dots \to & H_k(U_p) & \to & \check{H}_k(U_p \cup C(X)) & \to & \check{H}_{k-1}(X) & \to \dots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & B_n & \to & H_m(U_n) & \to \dots \to & H_k(U_n) & \to & \check{H}_k(U_n \cup C(X)) & \to & \check{H}_{k-1}(X) & \to \dots \end{array}$$

is commutative for $p > n$.

By applying Lemma 2.3, 2.4 and 2.5 we infer that $(\check{H}_k(U_n \cup C(X), \check{H}_k(1_n^{n+1}))$ satisfies the Mittag-Leffler condition for $k \leq m$, where 1_n^{n+1} is the inclusion map. Since $\varprojlim \check{H}_k(U_n \cup C(X) = \check{H}_k(C(X)) = 0$ (see Lemma 3.4), then

$(\check{H}_k(U_n \cup C(X), \check{H}_k(1_n^{n+1}))$ is isomorphic to the trivial group in pro-Gr for $k \leq m$ by a result of J. Keesling [20] (see also [13]). Hence for each n there exists $p > n$ such that the inclusion 1_n^p induces zero homomorphisms of Čech homology groups in dimensions less than or equal to m. Since the projection $U_n \cup C(X) \to U_n/X$ is a shape equivalence for each n, we get that the inclusion $U_p/X \to U_n/X$ induces zero homomorphisms on Čech homology groups up to dimension m. By a result in [12] the space Y/X is an LC^1-space and by the result of W. Hurewicz [18] the space Y/X is an LC^m-space.

Lemma 4.4. *Let* X *be a subcontinuum of the Hilbert cube* Q. *If* Q/X *is an* LC^m*-space* ($m \geq 1$), *then* X *is nearly* 1*-movable and pro-*$H_k(X)$ *is stable for* $k < m$ *and satisfies the Mittag-Leffler condition for* $k = m$.

Proof. By a result of N. Shrikhande [31] X is nearly 1-movable (see also [12]).

Take a decreasing sequence $(A_n)_{n=1}^{\infty}$ of ANR's in Q such that $X = \cap A_n$. Then we have the following exact sequence for each n

$$\to H_{k+1}(Q) \to H_{k+1}(Q \cup C(A_n)) \to H_k(A_n) \to H_k(Q) \to \ldots$$

i.e. the homomorphism from $H_{k+1}(Q \cup C(A_n))$ into $H_k(A_n)$ is an isomorphism for each k. Thus pro-$H_k(X)$ is isomorphic to $(H_{k+1}(Q \cup C(A_n), H_{k+1}(i_n^{n+1}))$, where i_n^{n+1} is the inclusion map. Since the projections $Q \cup C(A_n) \to Q/A_n$ are homotopy equivalences, pro-$H_k(X)$ is isomorphic to $(H_{k+1}(Q/A_n), H_{k+1}(p_n^{n+1}))$, where $p_n^{n+1}\colon Q/A_{n+1} \to Q/A_n$ is the natural projection.

Now $Q/X = \varprojlim (Q/A_n, p_n^{n+1})$ and therefore pro-$H_k(X)$ is isomorphic to pro-$H_{k+1}(Q/X)$. Since pro-$H_{k+1}(Q/X)$ is stable for $k < m$ and satisfies the Mittag-Leffler condition for $k = m$ (see Theorem 3.3), the proof of Lemma 4.4 is finished.

As an immediate consequence from Theorem 4.3, Lemma 4.2 and Lemma 4.4 we get

Theorem 4.5. *For a continuum* X *the following conditions are equivalent for* $n \geq 1$*:*

a) X *is an* LC^n*-divisor,*

b) Y/X *is an* LC^n*-space for some* LC^n*-space* Y *containing* X,

c) X *is nearly* 1*-movable and pro-*$H_k(X)$ *is stable for* $k < n$ *and satisfies the Mittag-Leffler condition for* $k = n$.

Corollary 4.6. *Being an* LC^n*-divisor is a hereditary shape invariant.*

Proof. If $Sh(Y) \leq Sh(X)$ and X is an LC^n-divisor, then by Theorem 4.5 X is nearly 1-movable and pro-$H_k(X)$ is stable for $k < n$ and satisfies the Mittag-Leffler condition for $k = n$. Since pro-$H_k(Y)$ is dominated by pro-$H_k(X)$ in pro-Gr for each k, then pro-$H_k(Y)$ is stable for $k < n$ and satisfies the Mittag-Leffler condition for $k = n$. Now Corollary 4.6 follows from Theorem 4.5 and the fact that being nearly 1-movable continuum is a hereditary shape invariant (see [25]).

Theorem 4.7. *Let* X *be a continuum such that* $Fd(X) = n < +\infty$. *Then the following conditions are equivalent:*

a) X *is an* ANR*-divisor,*

b) X *is an* LC^{n+1}*-divisor,*

c) X *is nearly* 1*-movable and pro-*$H_k(X)$ *is stable for* $k \leq n$.

Proof. It suffices to prove Theorem 4.7 for the case $\dim X = n$ (in view of [29] and Theorem 4.5).

a) $\to$ b) It follows from Lemma 4.2.

b) $\to$ a) By a result of Bothe [7] there is an ANR-space Y containing X such that $\dim Y \leq n+1$. Then Y/X is an LC^{n+1}-space and $\dim(Y/X) \leq n+1$. Hence Y/X is an ANR (see [5], p. 122) and by results of Hyman X is an ANR-divisor.

b) $\leftrightarrow$ c) It follows from Theorem 4.5 in view of the fact that $\text{pro-}H_{n+1}(X) = 0$.

Corollary 4.8. *In the class of continua of finite fundamental dimension the property of being an* ANR-*divisor is a hereditary shape invariant. In particular each* FANR-*space is an* ANR-*divisor.*

Proof. Analogous to the proof of Corollary 4.6.

Example 4.9. We construct an ANR-divisor X whose fundamental dimension is not finite.

For each n let $f_n\colon S^1 \vee S^n \to S^1 \vee S^n$ be a map such that $f_n/S^1 = \text{id}$ and $f_n/S^n\colon S^n \to S^1 \vee S^n$ is the composition of maps $g_n\colon S^n \to \bigvee_{i=1}^{\infty} S_i^n$ and $e_n\colon \bigvee_{n=1}^{\infty} S_i^n \to S^1 \vee S^n$, where $\pi_n(e_n)$ is an isomorphism and g_n represents the difference $[S_1^n] - [S_2^n]$ of two generators of $\pi_n(\bigvee_{i=1}^{\infty} S_i^n)$.

Then $H_k(f_n) = 0$ for each k and the induced map $f_n'\colon (S^1 \vee S^n)/S^1 \to (S^1 \vee S^n)/S^1$ is homotopically trivial.

Let $X_n = \bigvee_{k=1}^{n} S^k$ and let $h_n^{n+1}\colon X_{n+1} \to X_n$ be defined by $h_n^{n+1}(x) = f_k(x)$ for $x \in S^k$, $k \leq n$, and S^{n+1} is mapped onto the base point.

Let $X = \lim\limits_{\leftarrow} (X_n, h_n^{n+1})$. Then $S^1 \subset X$ and X/S^1 is an FAR. Since FAR's are ANR-divisors we infer by a result of Hyman [19] that X is an ANR-divisor.

Observe that Fd(X) is not finite because finiteness of Fd(X) would imply triviality of $\pi_n(f_n)^k$ for $n = Fd(X) + 1$ and some k.

Remark. Example 4.9 is constructed in the spirit of an example in [12].

Analogous to the corresponding results for ANR-divisors in [19] one can prove the following

Theorem 4.10. *Let* X *and* Y *be continua. If* X, Y *and* $X \cap Y$ *are* LC^n*-divisors, then* $X \cup Y$ *is an* LC^n*-divisor. If* $X \cup Y$ *and* $X \cap Y$ *are* LC^n*-divisors, then* X *and* Y *are* LC^n*-divisors. If* $X \subset Y$ *and* X *and* Y/X *are* LC^n*-divisors, then* Y *is an* LC^n*-divisor.*

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References

[1] S. Armentrout, *Decompositions and absolute neighborhood retracts*, Lecture Notes in Math. 438, Springer, New York, 1975.

[2] M. Artin and B. Mazur, *Etale homotopy theory*, Lecture Notes in Math. 100, Springer, 1969.

[3] M. F. Atyiah and G. Segal, *Equivariant* K-*theory and completion*, J. Diff. Geometry 3 (1969), 1-18.

[4] S. Bogatyi, *On a Vietoris theorem in the category of homotopies and a problem of Borsuk*, Fund. Math. 84 (1974), 209-228.

[5] K. Borsuk, *Theory of retracts*, Monografie Matematyczne 44, Warszawa, 1967.

[6] ________, *Theory of shape*, Monografie Matematyczne 59, Warszawa, 1975.

[7] H. Bothe, *Eine Einbettung* m-*dimensionaler Mengen in einen* (m+1)-*dimensionalen absoluten Retrakt*, Fund. Math. 51 (1962), 209-224.

[8] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math. 304 (1972), Springer, New York.

[9] Z. Čerin, C_p-*movable at infinity spaces, compact* ANR-*divisors and property* UVW^n, Publ. de l' Institut Math. 23 (1978), 53-65.

[10] T. A. Chapman, *On some applications of infinite-dimensional manifolds to the theory of shape*, Fund. Math. 76 (1972), 181-193.

[11] J. Dydak, *The Whitehead and Smale theorems in shape theory*, Dissertationes Mathematicae 156 (1979), 1-50.

[12] ________, *Some properties of nearly* 1-*movable continua*, Bull. Ac. Pol. Sci. 25 (1977), 685-689.

[13] ________, *An algebraic condition characterizing* FANR-*spaces*, Bull. Ac. Pol. Sci. 24 (1976), 501-503.

[14] D. A. Edwards and H. M. Hastings, *Čech and Steenrod homotopy theories with applications to geometric topology*, Lecture Notes in Math. 542, Springer, New York, 1976.

[15] R. Geoghegan, *A note on the vanishing of* $\lim^1$, (preprint).

[16] B. I. Gray, *Spaces of the same* n-*type for all* n, Topology 5 (1966), 241-243.

[17] S.-T. Hu, *Theory of retracts*, Wayne University Press, Detroit, 1965.

[18] W. Hurewicz, *Homotopie, Homologie, und lokaler Zusammenhang*, Fund. Math. 25 (1935), 467-485.

[19] D. M. Hyman, ANR-*divisors and absolute neighborhood contractibility*, Fund. Math. 62 (1968), 61-73.

[20] J. E. Keesling, *On the Whitehead theorem in shape theory*, Fund. Math. 92 (1976), 247-253.

[21] G. Kozlowski and J. Segal, *Local behavior and the Vietoris and Whitehead theorems in shape theory*, Fund. Math. 99 (1978), 210-219.

[22] S. Mardešic, *Shapes for topological spaces*, Gen. Top. Appl. 3 (1973), 265-282.

[23] ________ and J. Segal, *Shape of compacta and* ANR-*systems*, Fund. Math. 72 (1971), 41-59.

[24] S. Mardešic, *On the Whitehead theorem in shape theory II*, Fund. Math. 91 (1976), 93-103.

[25] D. R. McMillan, *One-dimensional shape properties and three manifolds*, Studies in Topology, Academic Press, 1975, pp. 367-381.

[26] J. Milnor, *On axiomatic homology theory*, Pacific J. Math. 12 (1966), 337-341.

[27] K. Morita, *On shapes of topological spaces*, Fund. Math. 86 (1975), 251-259.

[28] ________, *The Hurewicz and the Whitehead theorems in shape theory*, Sci. Rep. of the Tokyo Kyoiku Daigaku, Sec. A, 12 (1974), 246-258.

[29] S. Nowak, *Some properties of the fundamental dimension*, Fund. Math. 85 (1974), 211-227.

[30] R. H. Overton, *Čech homology for movable compacta*, Fund. Math. 77 (1973), 241-251.

[31] N. Shrikhande, *Homotopy properties of decomposition spaces*, Abstract 75T-638, Notices AMS, Apr. 1975.

[32] E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

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