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FACTORING OPEN SUBSETS OF R^∞ WITH CONTROL

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In this paper we show that if U is an open subset of R^∞ = dir lim R^n, then the projection map π: U $\times$ $R^\infty \to$ U can be approximated by homeomorphisms. One corollary of this result is that any homotopy equivalence f: U $\to$ V between open subsets of R^∞ is homotopic to a homeomorphism. A second corollary is that if h: $|K| \to |L|$ is a homotopy equivalence, where K and L are countable simplicial complexes, then f $\times$ id: $|K| \times R^\infty \to |L| \times R^\infty$ is homotopic to a homeomorphism.

1. Background and Statement of Results

Let R denote the reals, and let $R^\infty = \varinjlim R^n$. Let M and N denote paracompact, connected R^∞-manifolds. It is not known if M is stable, i.e. if M $\times$ R^∞ is homeomorphic to M. In [2] it is shown that (1) M $\times$ R^∞ embeds as an open subset of R^∞, and that (2) if M and N have the same homotopy type, then M $\times$ R^∞ and N $\times$ R^∞ are homeomorphic. Thus, if it were known that R^∞-manifolds are stable then it would follow that M embeds as an open subset of R^∞ and that if M and N have the same homotopy type then they are homeomorphic. In [4] it is shown that if U is an open subset of R^∞ then U $\times$ R^∞ is homeomorphic to U. Here we improve this result and show that there are homeomorphisms U $\times$ $R^\infty \to$ U arbitrarily close to the projection map (Theorem 1 below). Hopefully, the techniques used here will be useful in proving stability for general R^∞-manifolds. Note, too, that because of (1) above and Theorem 1 the existence of any homeomorphism $M \times R^\infty \to M$ will now imply the existence of homeomorphisms $M \times R^\infty \to M$ arbitrarily close to the projection map.

Theorem 1. *If* U *is an open subset of* R^∞ *and* $\mathcal{V}$ *is an open cover of* U, *then there is a homeomorphism* $g\colon U \times R^\infty \to U$ *which is* $\mathcal{V}$*-close to the projection map* $\pi\colon U \times R^\infty \to U$.

By g $\mathcal{V}$-close to π we mean that for every $(x,y) \in U \times R^\infty$ there is a $V \in \mathcal{V}$ such that $\{g(x,y), x = \pi(x,y)\} \subset V$. Thus, Theorem 1 says that the projection map can be approximated by homeomorphisms. The proof of Theorem 1, given in section 3, refines the argument in [4] using some of the techniques developed in [3]. It also uses a theorem from piecewise linear (p.l.) topology which we prove in section 2 as Theorem 2. This p.l. theorem seems to be known, but we could not find a proof in the literature.

Since R^∞ is locally convex, e.g. [2, Theorem IV.1], we may take the cover $\mathcal{V}$ in Theorem 1 to consist of convex sets. Any homeomorphism $g\colon U \times R^\infty \to U$ which is $\mathcal{V}$-close to π will then be homotopic to π via the straight line homotopy. Thus, we obtain the following.

Corollary 1. *For any open subset* U *of* R^∞ *there is a homeomorphism* $g\colon U \times R^\infty \to U$ *which is homotopic to the projection map.*

Now, let $f\colon U \to V$ be a homotopy equivalence where U and V are open subsets of R^∞. By [2, Theorem II.9 and Prop. III.1] there is a homeomorphism $h\colon U \times R^\infty \to V \times R^\infty$ which is homotopic to $f \times id$. By Corollary 1 there are homeomorphisms $h_V: V \times R^\infty \to V$ and $h_U: U \times R^\infty \to U$, each homotopic to the corresponding projection. It follows that $h_V h h_U^{-1}: U \to V$ is a homeomorphism homotopic to f. We have proved the following.

Corollary 2. *Any homotopy equivalence* $f: U \to V$ *between open subsets of* R^∞ *is homotopic to a homeomorphism.*

With regard to Corollary 2 we remark that although (nonempty) open subsets of R are not metrizable they do have the homotopy type of ANR's [2, Theorem II.10]. Thus, Corollary 2 holds as well if f is a weak homotopy equivalence.

As indicated at the end of [5], if K and L are countable simplicial complexes, then $|K| \times R^\infty$ and $|L| \times R^\infty$ are homeomorphic to open subsets of R^∞. Thus, as a special case of Corollary 2 we obtain the following.

Corollary 3. *If* K *and* L *are countable simplicial complexes, and if* $f: |K| \to |L|$ *is a homotopy equivalence, then* $f \times id: |K| \times R^\infty \to |L| \times R^\infty$ *is homotopic to a homeomorphism.*

2. Preliminary results

For convenience, if x is an element of a space X we will often write x for $\{x\}$. If (X,d) is a metric space, $C \subset X$ and $\varepsilon > 0$, then by $B(C,\varepsilon)$ we denote $\{x \in X \mid d(C,x)<\varepsilon\}$. Let $I = [0,1]$. If $H: X \times I \to Y$ is a homotopy define H_t, $t \in I$, by $H_t(x) = H(x,t)$. If $Y = X$, $H_0 = id$ and each H_t is a homeomorphism we say that H is an *ambient isotopy*. If H is also p.l. we say that H is a *p.l. ambient isotopy*.

Theorem 2. *Let* P *be a finite polyhedron of dimension* k. *Let* $H: P \times I \to R^n \subset R^{n+1}$, $n \geq 2k+1$, *be a homotopy such that* $H_0 = f$ *and* $H_1 = g$ *are* p.l. *embeddings. Then given* $\varepsilon > 0$ *there is a p.l. ambient isotopy* $A: R^{n+1} \times I \to R^{n+1}$ *such that (a)* $A_1 f = g$ *and (b) for every* $x \in R^{n+1}$ *either* $A(x \times I) = x$ *or* $A(x \times I) \subset B(H(p \times I), \varepsilon)$ *some* $p \in P$.

Proof. Let $\delta = \varepsilon/6$. Define $H': P \times I \to R^{n+1}$ by $H'(p,t) = (H(p,t), t\delta)$. Let $P_0 = P \times \{0,1\}$. Then H'/p_0 is a p.l. embedding. Let d be the usual metric on R^{n+1}. By [6, Theorem 5.4, p. 61] there is a p.l. embedding $G: P \times I \to R^{n+1}$ such that $d(G,H') < \delta$ and $G/p_0 = H'/p_0$. Note that $d(G,H) < 2\delta$ and $G_0 = f$. Let $\bar{g} = G_1 = (g, \delta)$.

Choose $\omega < \delta/2$ such that $d(x,y) < \omega$ implies $d(\bar{g}f^{-1}(x), \bar{g}f^{-1}(y)) < \delta/2$ for every $x,y \in f(P)$. Choose $\eta > 0$ such that for every $p \in P$

i) $\operatorname{diam}(G(p \times [0,\eta])) < \omega/2$ and

ii) $\operatorname{diam}(G(p \times [1-\eta,1])) < \omega/2$.

Choose $\gamma > 0$ such that $\gamma < \omega$ and (1) $d(G(P \times [\eta,1]), G_0(P)) > \gamma$ and (2) $d(G(P \times [0,1-\eta]), G_1(P)) > \gamma$. Let $U = \frac{\gamma}{2}$-neighborhood of $G_1(P)$. By an engulfing theorem of Bing [1, Theorem B, p. 8] (taking $L = \phi$, $C = G_1(P)$ in the notation of [1]) there is a p.l. ambient isotopy $F: R^{n+1} \times I \to R^{n+1}$ such that $F/(G_1(P) \times I) = \mathrm{id}$, for every $x \in R^{n+1}$ either $F(x \times I) = x$ or $F(x \times I) \subset B(G(p \times I), \gamma/2)$ some $p \in P$, and $G(P \times I) \subset F_1(U)$. Define $E: R^{n+1} \times I \to R^{n+1}$ by $E(x,t) = F_{1-t}F_1^{-1}(x)$. Then E is a p.l. ambient isotopy with the same properties as F except that the last condition becomes $E_1(G(P \times I)) \subset U$.

We proceed to show that $d(E_1 f, g) < 2\delta$. For $p \in P$

choose $\theta(p) \in P$ as follows. If $F(f(p) \times I) = f(p)$ take $\theta(p) = p$. Otherwise take $\theta(p) = q$ where q is any point P such that $E(f(p)\times I) \subset B(G(q\times I), \gamma/2)$. Then, in either case, $E(f(p)\times I) \subset B(G(\theta(p)\times I), \gamma/2)$. Since $f(p) = E_0 f(p) \in B(G(\theta(p)\times I), \gamma/2)$ there is a $t_0 \in I$ such that $d(f(p) = G_0(p), G(\theta(p), t_0)) < \gamma/2$. By (1) and then (i) above it follows that $t_0 < \eta$ and $d(G(\theta(p), t_0), G_0(\theta(p)) = f(\theta(p))) < \omega/2$. Thus $d(f(p), f(\theta(p)) < \gamma/2 + \omega/2 < \omega$ so that, by choice of ω, $d(\bar{g}(p), \bar{g}(\theta(p))) < \delta/2$. Choose $t_1 \in I$ such that $d(E_1 f(p), G(\theta(p), t_1)) < \gamma/2$. Then, since $E_1(G(P\times I)) \subset U$, $d(G(\theta(p), t_1), G_1(P)) < \gamma$. Applying (2) and then (ii) above we obtain $t_1 > 1 - \eta$ and $d(G(\theta(p), t_1), G_1(\theta(p)) = \bar{g}(\theta(p))) < \omega/2$. Thus $d(E_1 f(p), \bar{g}(\theta(p))) < \gamma/2 + \omega/2 < \omega$, and $d(E_1 f(p), \bar{g}(p)) \leq d(E_1 f(p), \bar{g}(\theta(p))) + d(\bar{g}(\theta(p)), \bar{g}(p)) < \omega + \delta/2 < \delta$. Therefore, $d(E_1 f, g) < 2\delta$.

By the unknotting theorem in [7, p. 111] (with P = L) there is a p.l. ambient isotopy $F: R^{n+1} \times I \to R^{n+1}$ such that $F_1 E_1 f = g$, $\operatorname{diam}(F(x\times I)) < 2\delta$ for every x, and $F(x\times I) = x$ for every x such that $d(x, E_1 f(p)) \geq 2\delta$. Define $A: R^{n+1} \times I \to R^{n+1}$ by $A_t = F_t E_t$. Then A is a p.l. ambient isotopy satisfying the conclusion of the theorem. To see that (b) holds in the case where $E(x\times I) = x$ and $F(x\times I) \neq x$, note first that $d(x, E_1 f(p)) < 2\delta$. Thus, $d(x, g(p) = H_1(p)) < 4\delta$ so that $A(x\times I) \subset B(H(p\times I), 6\delta = \varepsilon)$ as required.

3. Proof of Theorem 1

In addition to the notation introduced at the beginning of section 2 we will use the following. If $H: X \times I \to Y$ is a homotopy and $\mathcal{G}$ is an open cover of Y we say that H is *limited* by $\mathcal{G}$ if for each $x \in X$, $H(x \times I) \subset G$, some $G \in \mathcal{G}$. If $\mathcal{G}$ is an open cover of Y and X is any space then $X \times \mathcal{G} = \{X \times G | G \in \mathcal{G}\}$. If $\mathcal{G}$ is an open cover of Y and $A \subset Y$ then $A \cap \mathcal{G} = \{A \cap G | G \in \mathcal{G}\}$. We use d_k to denote the usual metric on R^k. If $A \subset R^k$ we denote by $\mathrm{Int}_k A$ the topological interior of A in R^k, and, for $\varepsilon > 0$, we denote by $B^k(A,\varepsilon)$ the set $\{x \in R^k | d_k(A,x) < \varepsilon\}$. We identify R^n with $R^n \times \{0\} \subset R^{n+1}$. In this way $R^\infty = \cup\{R^n | n = 1,2,3,...\}$. If $X \subset R^\infty$ we let $X^n = X \cap R^n$.

We will need two lemmas. The proof of the first is straightforward, and we omit the proof. Lemma 2 is proved in [3].

Lemma 1. *Let* C *be a compact subset of a locally compact metric space* (X,d). *Let* $\mathcal{G}$ *be a collection of open subsets of* X *whose union contains* C. *Then there is an* $\varepsilon > 0$ *such that for each* $x \in C$, $B(x,\varepsilon) \subset G$ *some* $G \in \mathcal{G}$.

Lemma 2. [3, *Lemma* 4]. *Let* $H\colon X \times I \to Y$ *be a homotopy where* X *is a compact metric space and* Y *is a metric space. Let* $\mathcal{G}$ *be an open cover of* Y *such that* H *is limited by* $\mathcal{G}$. *Then there is an* $\varepsilon > 0$ *such that for every* $x \in X$ *there is a* $G \in \mathcal{G}$ *such that* $B(H(x \times I),\varepsilon) < G$.

Proof of Theorem 1. For convenience, we may assume that U is connected. Using elementary reasoning (e.g. see [2, Prop. III.1 and Prop. III.2]), $U = \cup\{C_n | n = 4,5,6,7,...\}$ where $C_n \subset R^n$ is compact, $C_n \subset C_{n+1}$, and where a subset G of U is open in U iff $G \cap C_n$ is open in C_n, $n \geq 4$. In what follows "manifold" will be used only for a compact, p.l.

manifold, possibly with boundary. We observe that if K is any compact set and $K \subset W$ where W is open in R^n, then there is an n-manifold M such that $K \subset \text{Int}_W M \subset M \subset W$. Given $\varepsilon > 0$ and $n \geq 2$, let $D(n,\varepsilon) = \{x = (x_1,\ldots,x_n) | x \in R^n$ and $|x_i| \leq \varepsilon, i = 1,2,\ldots,n\}$.

Choose a 4-dimensional manifold M_2 such that $C_4 \subset M_4 \subset U^4$. By Lemma 1 there is an $\varepsilon_2 > 0$ such that for every $x \in M_2$, $B^8(x,2\varepsilon_2) \subset V^8$, some $V \in \mathcal{V}$. Choose a manifold M_3 of dimension 8 such that $[M_2 \times D(2,\varepsilon_2)] \cup C_8 \subset \text{Int}_8 M_3 \subset M_3 \subset U^8$. Choose $\rho_2 > 0$ such that $\rho_2 < \text{mim}\{1, d_8(M_2 \times E_2, R^\infty \setminus \text{Int}_8 M_3)\}$. Let $E_2 = D(2,\varepsilon_2)$ and $F_2 = D(2,2)$. Define a p.l. homeomorphism $h_2\colon M_2 \times E_2 \to M_2 \times F_2$ by $h_2(m,e) = (m,(2/\varepsilon_2)e)$. Let $F_3 = D(6,3)$, and define $i_2\colon M_2 \times E_2 \to M_3 \times F_3$ by $i_2(m,e) = (((m,e),0),0)$ and $\beta_2\colon M_2 \times F_2 \to M_3 \times F_3$ by $\beta_2(m,f) = ((m,0),(f,0))$. Define $H_2\colon M_2 \times E_2 \times I \to M_3 \times F_3$ by $H_2(m,e,t) = \left\{\begin{matrix} ((m,(1-2t)e),0), & t \in [0,1/2] \\ ((m,0),((2/\varepsilon_2)e,0)), & t \in [1/2,1] \end{matrix}\right\}$. Then, regarding H_2 as a map into $U^8 \times R^6$, H_2 is limited by $\mathcal{V}^8 \times R^6$. By Theorem 2 there is a p.l. ambient isotopy $A_3\colon R^{14} \times I \to R^{14}$ such that $(A_3)_1 i_2 = \beta_2 h_2$ and such that for every $x \in R^{14}$ either $A_3(x \times I) = x$ or $A_3(x \times I) \subset B^{14}(H_2((m,e) \times I),\delta_3)$ some $(m,e) \in M_2 \times E_2$. It follows that $(A_3)_t(M_3 \times F_3) = M_3 \times F_3$. Thus, we may regard A_3 as a p.l. ambient isotopy $A_3\colon M_3 \times F_3 \times I \to M_3 \times F_3$. As such A_3 is limited by $(M_3 \times F_3) \cap (\mathcal{V}^8 \times R^6)$. Set $g_2 = h_2$.

Suppose, inductively, that for $n \geq 3$ we have defined M_k, F_k, $2 \leq k \leq n$; E_{k-1}, $\beta_{k-1}\colon M_{k-1} \times F_{k-1} \to M_k \times F_k$, $g_{k-1}\colon M_{k-1} \times E_{k-1} \to M_{k-1} \times F_{k-1}$, $i_{k-1}\colon M_{k-1} \times E_{k-1} \to M_k \times F_k$, $3 \leq k \leq n$; $A_n\colon M_n \times F_n \times I \to M_n \times F_n$; and $\alpha_{k-2}\colon M_{k-2} \times E_{k-2} \to M_{k-1} \times E_{k-1}$, $3 \leq k \leq n$, (the condition on the α's being only for $n > 3$) such that M_k is a 2^k-dimensional manifold in U^{2^k}, $(M_{k-1} \times E_{k-1}) \cup C_{2^k} \subset \mathrm{Int}_{2^k} M_k$, $F_k = D(2^k-2,k)$, $\beta_{k-1}(m,f) = ((m,0),(f,0))$, $i_{k-1}(m,e) = ((m,e),0)$, g_{k-1} is a p.l. homeomorphism, $g_{k-1}\alpha_{k-2} = \beta_{k-2}g_{k-2}$ (this condition, again, only for $n > 3$), and such that A_n is a p.l. ambient isotopy limited by $(M_n \times F_n) \cap (V^{2^n} \times R^{2^n-2})$ with $(A_n)_1 i_{n-1} = \beta_{n-1}g_{n-1}$.

We proceed to construct M_{n+1}, F_{n+1}, E_n, β_n, g_n, α_{n-1}, i_n and A_{n+1} satisfying analogous conditions. Define $\beta_n\colon M_n \times F_n \to U^{2^{n+1}} \times R^{2^{n+1}-2}$ by $\beta_n(m,f) = ((m,0),(f,0))$. Then $\beta_n A_n$ is limited by $V^{2^{n+1}} \times R^{2^{n+1}-2}$. By Lemma 2 there is a $\gamma_n > 0$ such that for every $x \in M_n \times F_n$, $B^{2^{n+2}-2}(\beta_n A_n(x \times I), \gamma_n) \subset V^{2^{n+1}} \times R^{2^{n+1}-2}$. Choose $\varepsilon_n > 0$ such that $\varepsilon_n < \gamma_n$ and such that $M_n \times D(2^n-2,\varepsilon_n) \subset U^{2^{n+1}-2}$. Let $E_n = D(2^n-2,\varepsilon_n)$. Choose manifold M_{n+1} of dimension 2^{n+1} such that $[(M_n \times E_n) \cup C_{2^{n+1}}] \subset \mathrm{Int}_{2^{n+1}} M_{n+1} \subset M_{n+1} \subset U^{2^{n+1}}$. Choose $\rho_n > 0$ such that $\rho_n < \mathrm{mim}\{1, d_{2^{n+1}}(M_n \times E_n, R^\infty \backslash \mathrm{Int}_{2^{n+1}} M_{n+1}\}$. Let $F_{n+1} = D(2^{n+1}-2,n+1)$. Define a p.l. homeomorphism $h_n\colon M_n \times E_n \to M_n \times F_n$ by $h_n(m,e) = (m,(n/\varepsilon_n)e)$. Define $i_n\colon M_n \times E_n \to M_{n+1} \times F_{n+1}$, and $\alpha_{n-1}\colon M_{n-1} \times E_{n-1} \to M_n \times E_n$ by $i_n(m,e) = ((m,e),0)$ and $\alpha_{n-1}(m,e) = ((m,e),0)$. Note that $\beta_n(M_n \times E_n) \subset M_{n+1} \times F_{n+1}$ and consider the following diagram.

$$\begin{array}{ccccc}
M_{n-1} \times E_{n-1} & \xrightarrow{\alpha_{n-1}} & M_n \times E_n & & \\
\Big\downarrow g_{n-1} & & \Big\downarrow h_n & \searrow^{i_n} & \\
M_{n-1} \times F_{n-1} & \xrightarrow{\beta_{n-1}} & M_n \times F_n & \xrightarrow{\beta_n} & M_{n+1} \times F_{n+1}
\end{array}$$

Let $g_n = (A_n)_1 h_n$. Then g_n is a p.l. homeomorphism, and $g_n \alpha_{n-1} = (A_n)_1{}_{n-1} = \beta_{n-1} g_{n-1}$. Define H_n: $M_n \times E_n \times I \to M_{n+1} \times F_{n+1}$ by

$$H_n(m,e,t) = \begin{cases} ((m,(1-4t)e),0), & t \in [0,1/4] \\ ((m,0),([n(4t-1)/\varepsilon_n]e,0)), & t \in [1/4,1/2] \\ \beta_n A_n(h_n(m,e),2t-1), & t \in [1/2,1]. \end{cases}$$

Then H_n is a homotopy between the p.l. embeddings i_n and $\beta_n g_n$. Also, if π_{n+1}: $M_{n+1} \times F_{n+1} \to M_{n+1}$ is the projection, then $\pi_{n+1} H_n(x \times I)$ is contained in the γ_n-neighborhood of $\pi_{n+1}\beta_n A_n(h_n(x) \times I)$. Thus, by choice of γ_n, H_n is limited by $(M_{n+1} \times F_{n+1}) \cap (\mathcal{V}^{2^{n+1}} \times R^{2^{n+1}-2})$. By Lemma 2 there is a $\delta_{n+1} > 0$ such that $\delta_{n+1} \leq \rho_n$ and such that for every $x \in M_n \times E_n$, $B^{2^{n+2}-2}(H_n(x \times I), \delta_{n+1}) \subset V^{2^{n+1}} \times R^{2^{n+1}-2}$, some $V \in \mathcal{V}$. By Theorem 2 there is a p.l. ambient isotopy A_{n+1}: $R^{2^{n+2}-2} \times I \to R^{2^{n+2}-2}$ such that $(A_{n+1})_1 i_n = \beta_n g_n$ and such that for every $x \in R^{2^{n+2}-2}$ either $A_{n+1}(x \times I) = x$ or $A_{n+1}(x \times I) \subset B^{2^{n+2}-2}(H_n(m,e) \times I), \delta_{n+1})$ some $(m,e) \in M_n \times E_n$. It follows that $(A_{n+1})_t$ is the identity off $M_{n+1} \times F_{n+1}$, $t \in I$. Thus, we may regard A_{n+1} as a p.l. ambient isotopy A_{n+1}: $M_{n+1} \times F_{n+1} \times I \to M_{n+1} \times F_{n+1}$. As such, A_{n+1} is limited by $(M_{n+1} \times F_{n+1}) \cap (\mathcal{V}2^{n+1} \times R^{2n+2-2})$. This completes the inductive step.

By induction we have α_n, β_n, g_n, $n \geq 2$, such that the following diagram commutes for every n.

$$\begin{array}{ccc} M_n \times E_n & \xrightarrow{\alpha_n} & M_{n+1} \times E_{n+1} \\ \downarrow g_n & & \downarrow g_{n+1} \\ M_n \times F_n & \xrightarrow{\beta_n} & M_{n+1} \times F_{n+1} \end{array}$$

The g_n's induce a homeomorphism of direct limits,

$$g_\infty\colon \text{dir lim}\{M_n \times E_n;\alpha_n\} \to \text{dir lim}\{M_n \times F_n;\beta_n\}.$$

As shown in [4,p. 379] dir lim$\{M_n \times E_n;\alpha_n\}$ is homeomorphic to U and dir lim$\{M_n \times F_n;\beta_n\}$ is homeomorphic to $U \times R^\infty$. Thus, $(g_\infty)^{-1}$ induces a homeomorphism $g\colon U \times R^\infty \to U$. To see that g is $\mathcal{V}$-close to π, let $(m,x) \in U \times R^\infty$. Then $y = g(m,x) \in M_n$, some n, and $(m,x) = g_\infty(y) = g_n(y,0) \equiv \beta_n g_n(y,0) = ((m,0), (x,0))$. Since $\{(A_{n+1})_1 i_n((y,0)) = \beta_n g_n((y,0)) = ((m,0),(x,0)), (A_{n+1})_0 i_n((y,0)) = ((y,0),0)\} \subset V^{2^{n+1}} \times R^{2^{n+1}-2}$ some $V \in \mathcal{V}$, we have $\{(m,0),(y,0)\} \subset V^{2^{n+1}}$, some $V \in \mathcal{V}$, as required. The proof is now complete.

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