TOPOLOGY PROCEEDINGS

Volume 3, 1978

Pages 375–405

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN: 0146-4124

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1. Introduction

In the theory of covering properties of topological spaces, it has proved fruitful to search for characterizations of these properties by conditions different from and, if possible, weaker than those appearing in the original definitions. Perhaps the earliest example of this type of a characterization is the one given by Alexandroff and Urysohn for compactness: a topological space is compact if, and only if, every monotone open cover of the space has a finite subcover ([1]). The best known example, and perhaps the most important result in this field of research, is A. H. Stone's coincidence theorem: a Hausdorff space is paracompact if, and only if, the space is fully normal ([31]). A remarkable aspect of Stone's result is that it equates two properties which at a first glance appear to be completely dissimilar. In the proof of his theorem, Stone employed a very powerful technique that has later been modified to yield many characterizations of paracompactness and of related properties. Using a variation of this technique, E. Michael obtained a result that explains Stone's coincidence theorem by characterizing paracompactness (in the class of Hausdorff spaces) by a condition that is easily seen to be weaker than the conditions defining paracompactness and full normality ([23]). The applicability of Stone's technique of proof is not restricted to the theory of paracompact spaces: D. Burke has used modifications of this technique to show that the property of subparacompactness has characterizations analoguous to those given by Stone and Michael for paracompactness ([5]). Though the most important and successful so far, Stone's technique is not the only one available to attack problems in covering theory: J. Mack and W. Sconyers have taken important steps on the path initiated by Alexandroff and Urysohn by characterizing paracompactness and metacompactness in terms of refinements of monotone open covers ([20] and [27]).

In the present paper, we give several characterizations for some of the covering properties mentioned above. We obtain these characterizations by first studying submetacompact spaces. Submetacompact (= θ-refinable) spaces were introduced by J. M. Worrell Jr. and H. H. Wicke in [33]. These spaces have turned out to be useful in the theory of covering properties as well as in the theory of generalized metric spaces. Submetacompactness provides a simultaneous generalization of paracompactness, metacompactness and subparacompactness; on the other hand, as we will see in Section 2 below, simple expandability-type criteria allow us to pass from submetacompactness to these more restricted properties. Thus the theory of submetacompact spaces provides a unified approach to a large portion of covering theory.

In Section 3 of this paper we use an extension of Stone's technique of proof to obtain a characterization for submetacompact spaces by a condition significantly weaker than the one defining these spaces. We then use this characterization to exhibit some invariance properties of submetacompactness. In the end of the section we characterize the three covering properties related to submetacompactness.

In Section 4 we show that an Alexandroff-Urysohn type characterization holds also for submetacompactness. This allows us to restrict our consideration to refinements of interior-preserving open covers and to obtain the following result: a topological space is submetacompact if, and only if, every directed open cover of the space has a σ-closure-preserving closed refinement. As a corollary to this result, we see that a continuous image of a submetacompact space under a closed mapping is submetacompact.

In Section 5 we show that the strong $\Sigma^{\#}$-spaces of E. Michael are submetacompact. This result shows that the property of submetacompactness is of some independent interest and not just a useful stepping-stone to other, stronger properties. We also show that every orthocompact strict p-space is submetacompact and that a strict p-space is submetacompact if, and only if, it is a $\Sigma^{\#}$-space.

Notation and Terminology. Our terminology follows that of [13]; we do not, however, require paracompact or metacompact spaces to satisfy any separation axioms.

The set $\{1,2,\cdots\}$ of the natural numbers is denoted by $\mathbf{N}$. The symbol ϕ stands for a "0-tuple" and $\mathbf{N}^0 = \{\phi\}$. If $n \in \mathbf{N}$ and $(n_1,\cdots,n_k) \in \mathbf{N}^k$ for some $k = 0,1,\cdots$, then $(n_1,\cdots,n_k) \oplus n$ denotes the element $(n_1,\cdots,n_k,n)$ of the set $\mathbf{N}^{k+1}$. An infinite sequence whose n^{th} term is x_n (for $n \in \mathbf{N}$) is denoted by $\langle x_n \rangle_{n=1}^{\infty}$, or simply by $\langle x_n \rangle$.

Throughout the following, X denotes a topological space. Let $\mathcal{L}$ be a family of subsets of X. The symbol $\mathcal{L}^F$ is used to denote the family consisting of all finite unions of sets from $\mathcal{L}$. Note that $\mathcal{L}^F$ is a directed family (see [20]). Let A be a subset of X. The families $\{L \cap A \mid L \in \mathcal{L}\}$ and $\{L \in \mathcal{L} \mid L \cap A \neq \emptyset\}$ are denoted by $\mathcal{L}|A$ and $(\mathcal{L})_A$, respectively. If $A = \{x\}$, then we write $(\mathcal{L})_x$ in room of $(\mathcal{L})_A$. We let $St(A,\mathcal{L}) = \cup(\mathcal{L})_A$ and $St(A,\mathcal{L})^\circ = Int(St(A,\mathcal{L}))$.

Let $\mathcal{N}$ and $\mathcal{L}$ be families of subsets of X. If for each $N \in \mathcal{N}$, there is $L \in \mathcal{L}$ such that $N \subset L$, then $\mathcal{N}$ is a *partial refinement* of $\mathcal{L}$; if, moreover, $\cup\mathcal{N} = \cup\mathcal{L}$, then $\mathcal{N}$ is a *refinement* of $\mathcal{L}$. If $\mathcal{L}_1,\cdots,\mathcal{L}_n$ are covers of X, we let $\bigwedge_{i=1}^{n} \mathcal{L}_i = \{\bigcap_{i=1}^{n} L_i \mid L_i \in \mathcal{L}_i \text{ for each } i\}$; note that this family is a refinement of every $\mathcal{L}_i$. Let $\mathcal{N}$ and $\mathcal{L}$ be covers of X. $\mathcal{N}$ is a *point-star refinement of* $\mathcal{L}$ *at a point* $x \in X$ provided that $St(x,\mathcal{N}) \subset L$ for some $L \in \mathcal{L}$. A sequence $\langle \mathcal{N}_n \rangle$ of covers of X is a *point-star refining sequence* for the cover $\mathcal{L}$ provided that for each $x \in X$, there exists $n \in \mathbf{N}$ such that $\mathcal{N}_n$ is a point-star refinement of $\mathcal{L}$ at x.

A family $\mathcal{L}$ of subsets of X is *interior-preserving* if for each $\mathcal{L}' \subset \mathcal{L}$, we have $Int \cap \mathcal{L}' = \cap\{Int\ L \mid L \in \mathcal{L}'\}$. Note that $\mathcal{L}$ is interior-preserving iff the family $\{X \sim L \mid L \in \mathcal{L}\}$ is closure-preserving. Every point-finite family of subsets of X is interior-preserving. A family $\mathcal{U}$ of open subsets of X is interior-preserving iff for each $x \in X$, the set $\cap(\mathcal{U})_x$ is open. Consequently, if $\mathcal{U}$ is an interior-preserving open family, then the families $\{\cup\mathcal{U}' \mid \mathcal{U}' \subset \mathcal{U}\}$ and $\{\cap\mathcal{U}' \mid \mathcal{U}' \subset \mathcal{U}\}$ are interior-preserving and open; in particular, the family $\mathcal{U}^F$ is interior-preserving.

A cover $\mathcal{L}$ of X is *semi-open* if $x \in St(x,\mathcal{L})^\circ$ for each

$x \in X$. It is easily seen that $\mathcal{L}$ is a semi-open cover of X iff $\overline{A} \subset St(A, \mathcal{L})$ for each $A \subset X$. Every closure-preserving closed cover, as well as every open cover, is semi-open. Note that if $\mathcal{L}_1, \cdots, \mathcal{L}_n$ are semi-open covers of X, then the cover $\bigwedge_{i=1}^{n} \mathcal{L}_i$ is semi-open. For some other properties of semi-open and interior-preserving families, see [16].

2. Submetacompactness and Other Covering Properties

We have decided to use the name "submetacompact" for θ-refinable spaces, since this name makes evident the relation these spaces have to some other spaces defined by covering properties. To retain some continuity between our terminology and that concerning various generalizations of θ-refinable spaces, we adopt the following definition.

Definition 2.1. A sequence $\langle \mathcal{L}_n \rangle$ of covers of X is a θ-*sequence* if for each $x \in X$, there exists $n \in \mathbb{N}$ such that the family $\mathcal{L}_n$ is point-finite at x.

Remark. Worrell and Wicke observed in [33] that the set of points at which an open cover of X is point-finite is an F_σ-subset of X and they decuded the following: a cover $\mathcal{N}$ of X has a θ-sequence of open refinements iff there exists a countable closed cover $\mathcal{C}$ of X such that for each $C \in \mathcal{C}$, the cover $\mathcal{N}|C$ of the subspace C of X has a point-finite open (in C) refinement.

Definition 2.2. A topological space is *submetacompact* if every open cover of the space has a θ-sequence of open refinements.

We have the following diagram of implications between submetacompactness and some other covering properties.

$$\begin{array}{ccc} \text{paracompact} & \Rightarrow & \text{metacompact} \\ \Downarrow & & \Downarrow \\ \text{subparacompact} & \Rightarrow & \text{submetacompact} \end{array}$$

(The leftmost implication holds in the class of Hausdorff-spaces and the other three hold without any restrictions.)

It is well known that none of the implications in the above diagram is reversible; however, there are certain additional conditions under which these implications can be reversed. In [33], Worrell and Wicke extended earlier results of Michael and Nagami ([22] and [26]) and McAuley ([21]) with the following theorem: every collectionwise normal submetacompact Hausdorff-space is paracompact. Since paracompact Hausdorff-spaces are collectionwise normal, it follows from this theorem that collectionwise normality characterizes paracompactness in the class of submetacompact Hausdorff-spaces. We shall now indicate some other conditions that characterize stronger covering properties in the class of submetacompact spaces.

Recall that an *expansion* of a family $\mathcal{N}$ of subsets of X is a family $\{E(N) | N \in \mathcal{N}\}$ of subsets of X such that $N \subset E(N)$ for each $N \in \mathcal{N}$. The space X is *almost discretely expandable* provided that every discrete family $\mathcal{F}$ of closed subsets of X has an open expansion $\{V(F) | F \in \mathcal{F}\}$ such that for each $x \in X$, the family $\{F \in \mathcal{F} | x \in V(F)\}$ is finite ([30]; in [4], spaces with this property are called point-wise collectionwise normal). The space X is *θ-expandable* provided that every locally finite family $\mathcal{F}$ of closed subsets of X has a sequence $\langle \{V_n(F) | F \in \mathcal{F}\}\rangle_{n=1}^{\infty}$ of open expansions such that for each $x \in X$, there exists $n \in \mathbf{N}$ and a neighborhood W of x such that the family $\{F \in \mathcal{F} | V_n(F) \cap W \neq \emptyset\}$ is finite ([29]). The space X is *discretely subexpandable* provided that every discrete family $\mathcal{F}$ of closed subsets of X has a sequence $\langle\{V_n(F) | F \in \mathcal{F}\}\rangle_{n=1}^{\infty}$ of open expansion such that for each $x \in X$, there exists $n \in \mathbf{N}$ such that $x \in V_n(F)$ for at most one $F \in \mathcal{F}$ ([19]; this property is called collectionwise subnormality in [11]).

Theorem 2.3. *Let* X *be a submetacompact space. Then*

(i) X *is paracompact iff* X *is* θ*-expandable.*

(ii) X *is metacompact iff* X *is almost discretely expandable.*

(iii) X *is subparacompact iff* X *is discretely subexpandable.*

(i) follows from Corollary 2.10 and Theorem 1.5 of [29] (note that the assumption of regularity appearing in Theorem 1.5 of [29] is unnecessary, since expandable spaces are countably paracompact, by Theorem 2.8 of [30]). (ii) is proved in [4] (it also follows from Theorems 4.3 and 2.8 of [30], since submetacompact spaces are countably metacompact). (iii) is proved in [11] (for a related result, see [19], Theorem 3.2).

3. Submetacompactness and Semi-Open Covers

We need some terminology to state the results of this section.

Definition 3.1. Let $\mathcal{N}$ and $\mathcal{L}$ be covers of X. We say that $\mathcal{L}$ is a *point-star* $\dot{F}$*-refinement of* $\mathcal{N}$ *at a point* $x \in X$, if there exists a finite subfamily $\mathcal{N}'$ of $\mathcal{N}$ such that $x \in \cap\mathcal{N}'$ and $St(x,\mathcal{L}) \subset \cup\mathcal{N}'$; if $\mathcal{L}$ is a point-star $\dot{F}$-refinement of $\mathcal{N}$ at each point of X, then we say that $\mathcal{L}$ is a *point-star* $\dot{F}$-*refinement* of $\mathcal{N}$. A *point-star* $\dot{F}$-*refining sequence* for $\mathcal{N}$ is a sequence $\langle \mathcal{L}_n \rangle$ of covers of X such that for each $x \in X$, some member of the sequence is a point-star $\dot{F}$-refinement of $\mathcal{N}$ at x.

The dot over the F appearing in these definitions stands for the "centeredness" condition $x \in \cap\mathcal{N}'$; without this condition we arrive at point-star F-refinements, that is, point-star refinements of the directed cover $\mathcal{N}^F$.

Note that a point-wise W-refinement ([17]; this concept was introduced in [32]) of a cover is a point-star $\dot{F}$-refinement of the cover; in particular, a point-finite refinement of a cover is a point-star $\dot{F}$-refinement of the cover. Similarly we see that a θ-sequence of refinements of a cover is a point-star $\dot{F}$-refining sequence for the cover.

Theorem 3.2. *A topological space is submetacompact iff every open cover of the space has a point-star* $\dot{F}$-*refining sequence by semi-open covers of the space.*

Proof. Since every open cover is semi-open and every θ-sequence of refinements of a cover is a point-star $\dot{F}$-refining sequence for the cover, the condition is necessary. To prove sufficiency, assume that every open cover of X has a point-star $\dot{F}$-refining sequence by semi-open covers of X. Let $\mathcal{U}$ be an open cover of X. Represent $\mathcal{U}$ in the form $\mathcal{U} = \{U_\alpha | \alpha < \gamma\}$, where γ is an ordinal. We use induction on n to define open covers $\mathcal{V}_t = \{V_\beta(t) | \beta < 2\gamma\}$ of X for $t \in \bigcup_{n=0}^{\infty} \mathbf{N}^n$. Let $V_\alpha(\phi) = \emptyset$ and $V_{\gamma+\alpha}(\phi) = U_\alpha$ for each $\alpha < \gamma$; this defines the open cover $\mathcal{V}_\phi = \{V_\beta(\phi) | \beta < 2\gamma\}$. Let $n \in \mathbf{N}$ and assume that we have defined the open covers $\mathcal{V}_s = \{V_\beta(s) | \beta < 2\gamma\}$ for $s \in \mathbf{N}^{n-1}$. For each $s \in \mathbf{N}^{n-1}$, the open cover $\mathcal{V}_s$ of X has a point-star $\dot{F}$-refining sequence $\langle \mathcal{L}_{s\oplus k} \rangle$ by semi-open covers of X. We now define, for all $s \in \mathbf{N}^{n-1}$, $k \in \mathbf{N}$ and $\alpha < \gamma$:

$$V_\alpha(s\oplus k) = U_\alpha \cap [V_\alpha(s) \cup St(X \sim \bigcup_{\beta \neq \gamma+\alpha} V_\beta(s), \mathcal{L}_{s\oplus k})^\circ]$$

(*) and

$$V_{\gamma+\alpha}(s\oplus k) = U_\alpha \cap (\bigcup_{\beta > \gamma+\alpha} V_\beta(s)) \cap St(X \sim \bigcup_{\beta < \gamma+\alpha} V_\beta(s), \mathcal{L}_{s\oplus k})^\circ.$$

Since every element of $\mathbf{N}^n$ can be uniquely represented in the form $s\oplus k$ for some $s \in \mathbf{N}^{n-1}$ and $k \in \mathbf{N}$, the above formulas define the families $\mathcal{V}_t = \{V_\beta(t) | \beta < 2\gamma\}$ for $t \in \mathbf{N}^n$. To complete the induction, it remains to verify that the families $\mathcal{V}_t$, $t \in \mathbf{N}^n$, are open covers of X. It is easily seen that these families consist of open subsets. To show that they cover X, let $t = s\oplus k$ be a member of $\mathbf{N}^n$ and let $x \in X$. Since the family $\mathcal{V}_s$ covers X, the set $\{\beta < 2\gamma | x \in V_\beta(s)\}$ is non-empty; let δ be the least element of this set. If $\delta < \gamma$, then $V_\delta(s) \subset V_\delta(s\oplus k)$ and hence $x \in V_\delta(s\oplus k)$. Assume that $\delta \geqq \gamma$. Let $\rho = \delta - \gamma$ so that $\delta = \gamma + \rho$. We show that either $x \in V_\rho(s\oplus k)$ or $x \in V_{\gamma+\rho}(s\oplus k)$. Assume that $x \notin V_{\gamma+\rho}(s\oplus k)$. Since $x \in V_{\gamma+\rho}(s)$, we have $x \in U_\rho$. By the definition of δ, we have $x \notin \bigcup_{\beta<\gamma+\rho} V_\beta(s)$. Consequently, $St(x, \mathcal{L}_{s\oplus k}) \subset St(X \sim \bigcup_{\beta<\gamma+\delta} V_\beta(s), \mathcal{L}_{s\oplus k})$. Since the cover $\mathcal{L}_{s\oplus k}$ is semi-open, it follows that $x \in St(X \sim \bigcup_{\beta<\gamma+\delta} V_\beta(s), \mathcal{L}_{s\oplus k})^\circ$. Since $x \notin V_{\gamma+\rho}(s\oplus k)$ and $x \in U_\rho \cap St(X \sim \bigcup_{\beta<\gamma+\delta} V_\beta(s), \mathcal{L}_{s\oplus k})^\circ$, it follows from the definition of the set $V_{\gamma+\rho}(s\oplus k)$ that $x \notin \bigcup_{\beta>\gamma+\rho} V_\beta(s)$. Consequently, $x \notin \bigcup_{\beta\neq\gamma+\rho} V_\beta(s)$. It follows that $x \in St(X \sim \bigcup_{\beta\neq\gamma+\rho} V_\beta(s), \mathcal{L}_{s\oplus k})^\circ$. Since $x \in U_\rho$, it follows that $x \in V_\rho(s\oplus k)$.

We have shown that if $x \notin V_{\gamma+\rho}(s\oplus k)$, then $x \in V_\rho(s\oplus k)$. Consequently, either $x \in V_\rho(s\oplus k)$ or $x \in V_{\gamma+\rho}(s\oplus k)$. It follows from the foregoing that the family $\mathcal{V}_{s\oplus k}$ covers X.

We have now defined the covers $\mathcal{V}_t$, $t \in \bigcup_{n=0}^{\infty} \mathbf{N}^n$. Note that there are only countably many of these covers and that each one of them is an open refinement of $\mathcal{U}$. To complete the proof, it suffices to show that for each $x \in X$, there exists $s \in \bigcup_{n=0}^{\infty} \mathbf{N}^n$ such that the family $\mathcal{V}_s$ is point-finite at x. Let $x \in X$. There exists a sequence $\langle k_n\rangle_{n=1}^{\infty}$ of natural numbers such that if we let $s(0) = \phi$ and $s(n) = (k_1,\cdots,k_n)$ for each $n \in \mathbf{N}$, then the cover $\mathcal{L}_{s(n)}$ is a point-star $\dot{F}$-refinement of $\mathcal{V}_{s(n-1)}$ for each $n \in \mathbf{N}$. For each $n \in \mathbf{N}$, let A_{n-1} be a finite subset of $\{\beta|\beta < 2\gamma\}$ such that $x \in \cap\{V_\beta(s(n-1))|\beta \in A_{n-1}\}$ and $St(x,\mathcal{L}_{s(n)}) \subset \cup\{V_\beta(s(n-1))|\beta \in A_{n-1}\}$, and let $\beta(n-1)$ be the largest element of the set A_{n-1}. Denote by δ the least element of the set $\{\beta(n-1)|n \in \mathbf{N}\}$ and let $m \in \mathbf{N}$ be such that $\delta = \beta(m-1)$. We show that the family $\mathcal{V}_{s(m)}$ is point-finite at x and we start by showing that $x \notin \bigcup_{\beta\geq\gamma} V_\beta(s(m))$. Since $\delta = \max A_{m-1}$ and $St(x,\mathcal{L}_{s(m)}) \subset \cup\{V_\beta(s(m-1))|\beta \in A_{m-1}\}$, we have $x \notin St(X \sim \bigcup_{\beta<\rho} V_\beta(s(m-1)),\mathcal{L}_{s(m)})$ whenever $\rho > \delta$. It follows, using the definition of the sets $V_\rho(s(m))$ for $\rho \geq \gamma$, that $x \notin \cup\{V_\rho(s(m))|\rho \geq \gamma$ and $\rho > \delta\}$. Consequently, to show that $x \notin \bigcup_{\beta\geq\gamma} V_\beta(s(m))$, it suffices to show that $\delta < \gamma$. Assume on the contrary that $\delta \geq \gamma$. Then it follows from the foregoing that $x \notin \bigcup_{\rho>\delta} V_\rho(s(m))$. Since $\delta \geq \gamma$ and $s(m+1) = s(m) \oplus k_{m+1}$, we have $V_\beta(s(m+1)) \subset \bigcup_{\rho>\beta} V_\rho(s(m))$ for each $\beta \geq \delta$. It follows that $x \notin \bigcup_{\beta\geq\delta} V_\beta(s(m+1))$. Since $x \in V_{\beta(m+1)}(s(m+1))$, it follows that $\beta(m+1) < \delta$. However, by the definition of

δ, we have $\beta(m+1) \geqq \delta$. This contradiction shows that we must have $\delta < \gamma$. It follows that $x \notin \bigcup_{\beta \geqq \gamma} V_\beta(s(m))$. Hence, to show that the family $\mathcal{V}_{s(m)}$ is point-finite at x, it suffices to show that the set $\{\alpha < \gamma | x \in V_\alpha(s(m))\}$ is finite. Let $C = \{\alpha < \gamma | x \in V_\alpha(s(m))\}$ and for each $n \leqq m$, let $B_n = \{\alpha < \gamma | \gamma + \alpha \in A_n\}$. Since the sets A_n are finite, so are the sets B_n; hence to show that the set C is finite, it suffices to show that $C \subset \bigcup_{n=1}^{m} B_n$. Let $\alpha \in C$. We have $x \in V_\alpha(s(m)) \sim V_\alpha(\phi)$ and hence there exists $k < m$ such that $x \in V_\alpha(s(k+1)) \sim V_\alpha(s(k))$. Since $\alpha < \gamma$, it follows using the definition of the set $V_\alpha(s(k+1))$ that $x \in St(X \sim \bigcup_{\beta \neq \gamma+\alpha} V_\beta(s(k)), \mathcal{L}_{s(k+1)})$. Let $y \in X \sim \bigcup_{\beta \neq \gamma+\alpha} V_\beta(s(k))$ be such that $x \in St(y, \mathcal{L}_{s(k+1)})$. Then $y \in St(x, \mathcal{L}_{s(k+1)}) \subset \cup\{V_\beta(s(k)) | \beta \in A_k\}$. Hence there exists $\nu \in A_k$ such that $y \in V_\nu(s(k))$. Since $y \in X \sim \bigcup_{\beta \neq \gamma+\alpha} V_\beta(s(k))$, we have $\nu = \gamma + \alpha$. Consequently, $\gamma + \alpha \in A_k$ and $\alpha \in B_k$. We have shown that $C \subset \bigcup_{n=1}^{m} B_n$. It follows that the set C is finite and the family $\mathcal{V}_{s(m)}$ is point-finite at x.

Before exhibiting some corollaries to the above theorem, we make two observations concerning the above proof; these observations will be used in the next section in the proofs of some further characterizations of submetacompactness.

Remark 3.2.1. Let $\underline{k}$ be a cardinal number. We say that a space X is $\underline{k}$-*submetacompact* if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leqq \underline{k}$ has a θ-sequence of open refinements. It follows from the above proof that X is $\underline{k}$-submetacompact if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leqq \underline{k}$ has a point-star $\dot{F}$-refining sequence by semi-open covers.

Remark 3.2.2. If the covers $\mathcal{U}$ and $\mathcal{L}_t$, $t \in \bigcup_{n=1}^{\infty} \mathbf{N}^n$, appearing in the above proof are interior-preserving and open, then it is easily seen that the covers $\mathcal{V}_t$, $t \in \bigcup_{n=0}^{\infty} \mathbf{N}^n$, defined by formulas (*) above, will also be interior-preserving and open. Hence it follows from the proof that if every interior-preserving open cover of X has a point-star $\dot{F}$-refining sequence by interior-preserving open covers of X, then every interior-preserving open cover of X has a θ-sequence of (interior-preserving and) open refinements.

The following is an immediate consequence of the result of Theorem 3.2.

Corollary 3.3. *A topological space is submetacompact iff every open cover of the space has a* θ*-sequence of semi-open refinements.*

The next two results deal with the preservation of the property of submetacompactness in certain topological operations.

Proposition 3.4. *A continuous image of a submetacompact space under a pseudo-open finite-to-one mapping is submetacompact.*

Proof. Let f be a continuous, pseudo-open finite-to-one mapping from a submetacompact space X onto a space Y. To show that Y is submetacompact, let $\mathcal{O}$ be an open cover of Y. Then the family $\mathcal{U} = \{f^{-1}(O) \mid O \in \mathcal{O}\}$ is an open cover of X. Let $\langle \mathcal{V}_n \rangle$ be a θ-sequence of open refinements of the cover $\mathcal{U}$. For each $n \in \mathbf{N}$, let $\mathcal{W}_n = \bigwedge_{k=1}^{n} \mathcal{V}_k$ and let $\mathcal{L}_n = \{f(W) \mid W \in \mathcal{W}_n\}$. It is easily seen (and follows from Lemma 1.4 of [18]) that for every $n \in \mathbf{N}$, the family $\mathcal{L}_n$ is a semi-open cover of Y. We show that $\langle \mathcal{L}_n \rangle$ is a point-star $\dot{F}$-refining sequence for $\mathcal{O}$. Let $y \in Y$ and let $f^{-1}\{y\} = \{x(1),\cdots,x(j)\}$. For each $i = 1,\cdots,j$ there exists $n(i) \in \mathbf{N}$ such that the family $(\mathcal{V}_{n(i)})_{x(i)}$ is finite. Let $\mathcal{V}' = \cup\{(\mathcal{V}_{n(i)})_{x(i)} | i=1,\cdots,j\}$ and for each $V \in \mathcal{V}'$, let $O(V) \in \mathcal{O}$ be such that $V \subset f^{-1}(O(V))$. Let $m = \max\{n(1),\cdots,n(j)\}$ and let $\mathcal{O}' = \{O(V) | V \in \mathcal{V}\}$. Then $\mathcal{O}'$ is a finite subfamily of $\mathcal{O}$ and $y \in \cap\mathcal{O}'$. It is easily seen that $St(f^{-1}\{y\},\mathcal{W}_m) \subset \cup\{f^{-1}(O) | O \in \mathcal{O}'\}$. It follows that $St(y,\mathcal{L}_m) \subset \cup\mathcal{O}'$. Hence the family $\mathcal{L}_m$ is a point-star $\dot{F}$-refinement of the cover $\mathcal{O}$ at point y. We have shown that $\langle \mathcal{L}_n \rangle$ is a point-star $\dot{F}$-refining sequence for $\mathcal{O}$. It follows from the foregoing and Theorem 3.2 that the space Y is submetacompact.

Proposition 3.5. *A topological space is submetacompact if it has a point-finite semi-open cover such that every set of the cover is contained in some submetacompact subspace of the space.*

Proof. To prove this result, use Lemma 1.3 of [18] and arguments similar to those used in the proof of the preceding proposition to produce a point-star $\dot{F}$-refining sequence by semi-open covers for a given open cover of the space; compare also with the proof of Corollary 2.3 of [18]. We omit the details.

Since every locally finite closed cover of a topological space is semi-open, it follows from Proposition 3.5 that the Locally Finite Sum Theorem holds for submetacompactness.

In the remaining results of this section we characterize the three covering properties related to submetacompactness through expandability conditions (Theorem 2.3).

Theorem 3.6. *A topological space is metacompact iff every open cover of the space has a semi-open point-star* $\dot{F}$*-refinement.*

Proof. Necessity is immediate. To prove sufficiency, assume that every open cover of X has a semi-open point-star $\dot{F}$-refinement. It follows from Theorem 3.2 that X is submetacompact. To show that X is metacompact it suffices, by Theorem 2.3, to show that X is almost discretely expandable. Let $\mathcal{F}$ be a discrete family of closed subsets of X. For each $F \in \mathcal{F}$, let $U(F) = X \sim \cup(\mathcal{F} \sim \{F\})$. Then the family $\{U(F) | F \in \mathcal{F}\}$ is an open cover of X. Let $\mathcal{L}$ be a semi-open point-star $\dot{F}$-refinement of $\mathcal{U}$. For each $F \in \mathcal{F}$, let $V(F) = St(F,\mathcal{L})^{\circ}$. Then the family $\{V(F) | F \in \mathcal{F}\}$ is an open expansion of the family $\mathcal{F}$. We show that for each $x \in X$, the family $\{F \in \mathcal{F} | x \in V(F)\}$ is finite. Let $x \in X$. Since $\mathcal{L}$ is a point-star $\dot{F}$-refinement of $\mathcal{U}$, there exists a finite subfamily $\mathcal{F}'$ of $\mathcal{F}$ such that $St(x,\mathcal{L}) \subset \cup\{U(F) | F \in \mathcal{F}'\}$. From the definition of the sets $U(F)$, $F \in \mathcal{F}$, it follows that we have $St(x,\mathcal{L}) \cap F = \emptyset$ for every $F \in \mathcal{F} \sim \mathcal{F}'$. Consequently, the family $\{F \in \mathcal{F} | x \in V(F)\}$ is contained in the finite family $\mathcal{F}'$. We have shown that X is almost discretely expandable.

Note that Theorem 3.6 gives as direct corollaries the characterizations of metacompactness given in [32] and [18].

Theorem 3.7. *A topological space is paracompact if every open cover of the space has a semi-open point-star refinement.*

Proof. Assume that every open cover of X has a semi-open point-star refinement. It follows from Theorem 3.6 that X is metacompact. We show that X is collectionwise normal; it then follows from the Michael-Nagami theorem that X is paracompact. Let $\mathcal{F}$ be a discrete family of closed subsets of X. For each $F \in \mathcal{F}$, let $U(F) = X \sim \cup(\mathcal{F} \sim \{F\})$. The family $\mathcal{U} = \{U(F) | F \in \mathcal{F}\}$ is an open cover of X. Let $\mathcal{L}$ be a semi-open point-star refinement of $\mathcal{U}$ and for each $F \in \mathcal{F}$, let $V(F) = St(F,\mathcal{L})^{\circ}$. The family $\{V(F) | F \in \mathcal{F}\}$ is an open expansion of the family $\mathcal{F}$ and it is easily seen that if F and F' are two distinct elements of $\mathcal{F}$, then $V(F) \cap V(F') = \emptyset$. We have shown that X is collectionwise normal.

By Corollary 3.5 of [16], an open cover of a topological space has a semi-open point-star refinement iff the cover has a cushioned refinement (this follows also from Lemma 3.10 below); hence Theorem 3.7 is just a reformulation of a well-known result of E. Michael ([23])).

Theorem 3.8. *A topological space is subparacompact iff every open cover of the space has a point-star refining sequence by semi-open covers of the space.*

Proof. Necessity follows from Theorem 1.2 of [5]. To prove sufficiency, assume that every open cover of X has a point-star refining sequence by semi-open covers of X. It follows from Theorem 3.2 that X is submetacompact. We show that X is discretely subexpandable; it then follows from Theorem 2.3 that X is subparacompact. Let $\mathcal{F}$ be a discrete family of closed subsets of X and let $U(F) = X \sim \cup(\mathcal{F} \sim \{F\})$

for each $F \in \mathcal{F}$. Let $\langle \mathcal{L}_n \rangle$ be a point-star refining sequence for the open cover $\{U(F) | F \in \mathcal{F}\}$ of X such that for each $n \in \mathbb{N}$, the family $\mathcal{L}_n$ is a semi-open cover of X. Setting $V_n(F) = St(F, \mathcal{L}_n)^\circ$ for all $F \in \mathcal{F}$ and $n \in \mathbb{N}$, we get a sequence $\langle \{V_n(F) | F \in \mathcal{F}\} \rangle_{n=1}^{\infty}$ of open expansions of $\mathcal{F}$. This sequence has the required property since if $x \in X$, then $St(x, \mathcal{L}_n) \subset U(F)$ for some $n \in \mathbb{N}$ and $F \in \mathcal{F}$; clearly $x \notin V_n(F')$ if $F' \neq F$.

Note that we could have proved the above result more directly, without relying on Theorem 2.3, by using the remark made after Definition 2.1 and the following observation: if $\mathcal{N}$ is a point-finite family of subsets of X and $\mathcal{L}$ a semi-open cover of X, and if we let $H(N) = \{x \in X | St(x, \mathcal{L}) \subset N\}$ for each $N \in \mathcal{N}$, then the family $\{H(N) | N \in \mathcal{N}\}$ is locally finite and $\overline{H(N)} \subset N$ for each $N \in \mathcal{N}$.

D. Burke showed in [6] that a perfect space is subparacompact if every open cover of the space has a σ-cushioned refinement and he asked whether this result remains true without the assumption of perfectness (see also Problem 2.8 of [19]). We now prove a lemma which can be used together with Theorem 3.8 to answer Burke's question.

Recall that a family $\mathcal{M}$ of subsets of X is *cushioned* in a family $\mathcal{N}$ of subsets of X provided that there exists a map $\Phi: \mathcal{M} \to \mathcal{N}$ such that $\overline{\cup \mathcal{M}'} \subset \cup \Phi(\mathcal{M}')$ for each $\mathcal{M}' \subset \mathcal{M}$ ([23]).

Lemma 3.9. *Let* $\mathcal{U}$ *be an open cover of* X *and let* A *be a subset of* X. *Then there exists a family of subsets of* X *covering* A *which is cushioned in* $\mathcal{U}$ *iff there exists a semi-open cover of* X *which is a point-star refinement of* $\mathcal{U}$ *at each point of* A.

Proof. Assume that there exists a family $\mathcal{M}$ of subsets of X such that $A \subset \cup\mathcal{M}$ and a map $\Phi: \mathcal{M} \to \mathcal{U}$ such that $\overline{\cup\mathcal{M}'} \subset \cup\Phi(\mathcal{M}')$ for each $\mathcal{M}' \subset \mathcal{M}$. For each $x \in \cup\mathcal{M}$, let $M_x \in (\mathcal{M})_x$ and let $V_x = \Phi(M_x)$. If we set $V_x = X$ for each $x \in X \sim \cup\mathcal{M}$, then it is easily seen that for each $B \subset X$, we have $\overline{B} \subset \cup\{V_x | x \in B\}$. It follows that for each $x \in X$, the set $W_x = X \sim \{y \in X | x \notin V_y\}$ is a neighborhood of x. Let $\mathcal{L} = \{\{x,y\} \subset X | x \in V_y$ and $y \in V_x\}$. Then $\mathcal{L}$ is a cover of X and for each $x \in X$, we have $St(x,\mathcal{L}) \subset V_x$. We have $V_x \in \mathcal{U}$ for each $x \in A$ and it follows that $\mathcal{L}$ is a point-star refinement of $\mathcal{U}$ at each point of A. It is easily verified that for each $x \in X$, the neighborhood $V_x \cap W_x$ of x is contained in the set $St(x,\mathcal{L})$; hence $\mathcal{L}$ is a semi-open cover of X.

To prove the converse, assume that there exists a semi-open cover $\mathcal{K}$ of X such that $\mathcal{K}$ is a point-star refinement of $\mathcal{U}$ at each point of A. Let $N(U) = \{x \in X | St(x,\mathcal{K}) \subset U\}$ for each $U \in \mathcal{U}$. Then the family $\mathcal{N} = \{N(U) | U \in \mathcal{U}\}$ of subsets of X covers the set A. For each $M \in \mathcal{N}$, let $\Phi(M) \in \mathcal{U}$ be such that $M = N(\Phi(M))$. Note that for each $M \in \mathcal{N}$, we have $St(M,\mathcal{K}) \subset \Phi(M)$. We show that $\overline{\cup\mathcal{N}'} \subset \cup\Phi(\mathcal{N}')$ for each $\mathcal{N}' \subset \mathcal{N}$. Let $\mathcal{N}' \subset \mathcal{N}$. Then $\overline{\cup\mathcal{N}'} \subset St(\cup\mathcal{N}',\mathcal{K})$ since the cover $\mathcal{L}$ is semi-open. We have $St(\cup\mathcal{N}',\mathcal{K}) = \cup\{St(N,\mathcal{K}) | N \in \mathcal{N}'\} \subset \cup\Phi(\mathcal{N}')$. Hence $\overline{\cup\mathcal{N}'} \subset \cup\Phi(\mathcal{N}')$.

Letting A = X in the above result, we have the result of Corollary 3.5 of [16].

A σ-*cushioned* refinement of a cover $\mathcal{N}$ of X is a cover $\bigcup_{n\in\mathbf{N}} \mathcal{M}_n$ of X such that for each $n \in \mathbf{N}$, the family $\mathcal{M}_n$ is cushioned in the cover $\mathcal{N}$. It follows from Lemma 3.9 that an open cover of X has a σ-cushioned refinement iff the cover has a point-star refining sequence by semi-open covers of X. Hence we can restate the result of Theorem 3.8 in the following way: a topological space is subparacompact iff every open cover of the space has a σ-cushioned refinement.

4. Submetacompactness and Directed Covers

In the last section we characterized submetacompactness, as well as some other covering properties, in terms of refinements of arbitrary open covers of a topological space. In this section we obtain some characterizations of submetacompactness in terms of refinements of some special open covers. We start with the following result.

Proposition 4.1. *A topological space is submetacompact iff every monotone open cover of the space has a θ-sequence of open refinements.*

Proof. Necessity is trivial. To prove sufficiency, assume that every monotone open cover of X has a θ-sequence of open refinements. We use transfinite induction and Remark 3.2.1 to show that X is $\underline{m}$-submetacompact for every cardinal number $\underline{m}$. There is nothing to prove for $\underline{m}$ finite. Assume that $\underline{m}$ is an infinite cardinal such that we have shown X to be $\underline{k}$-submetacompact for every $\underline{k} < \underline{m}$. To show that X is $\underline{m}$-submetacompact, it suffices to show that every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leq \underline{m}$ has a point-star $\dot{F}$-refining sequence by open covers of X. Let $\mathcal{U}$ be an open cover of X such that $|\mathcal{U}| \leq \underline{m}$. We can represent $\mathcal{U}$ in the form $\mathcal{U} = \{U(\alpha) | \alpha < \gamma\}$, where γ is the initial ordinal corresponding to $\underline{m}$. For each $\alpha < \gamma$, let $V(\alpha) = \bigcup_{\beta \leq \alpha} U(\beta)$. Then the family $\{V(\alpha) | \alpha < \gamma\}$ is a monotone open cover of X and hence this family has a θ-sequence, say $\langle \mathcal{V}_n \rangle$, of open refinements. We may assume that for each $n \in \mathbf{N}$, the family $\mathcal{V}_n$ is of the form $\mathcal{V}_n = \{V_n(\alpha) \mid \alpha < \gamma\}$, where $V_n(\alpha) \subset V(\alpha)$ for each $\alpha < \gamma$. For all $n \in \mathbf{N}$ and $\alpha < \gamma$, the family $\{U(\beta) \mid \beta \leq \alpha\} \cup \{\bigcup_{\beta > \alpha} V_n(\beta)\}$ is an open cover of X with cardinality less than $\underline{m}$; hence this family has a θ-sequence, say $\langle \mathcal{W}_{n,k}(\alpha) \rangle_{k=1}^{\infty}$, of open refinements.

For all $n \in \mathbf{N}$ and $h \in \mathbf{N}$, denote by $F_{n,h}$ the closed set $\{x \in X \mid |(\mathcal{V}_n)_x| \leq h\}$. For each $n \in \mathbf{N}$, let $H_n = \bigcup_{h \in \mathbf{N}} F_{n,h}$ and for each $x \in H_n$, denote by $\alpha(x,n)$ the largest element of the finite set $\{\alpha < \gamma \mid x \in V_n(\alpha)\}$. Note that if $n \in \mathbf{N}$ and $x \in H_n$, then $x \notin \bigcup_{\beta > \alpha(x,n)} V_n(\beta)$ and it follows that for each $k \in \mathbf{N}$, the family $(\mathcal{W}_{n,k}(\alpha(x,n)))_x$ is a partial refinement of the subfamily $\{U(\beta) \mid \beta \leq \alpha(x,n)\}$ of $\mathcal{U}$. For all $n \in \mathbf{N}$ and $k \in \mathbf{N}$, and for every $x \in H_n$, let $W_{n,k}(x)_k \in (\mathcal{W}_{n,k}(\alpha(x,n)))_x$, and further, let $O_{n,k}(x) = V_n(\alpha(x,n)) \cap [\bigcap_{i=1} W_{n,i}(x)]$. Then, for all $n \in \mathbf{N}$ and $k \in \mathbf{N}$, the family $\mathcal{O}_{n,k} = \{O_{n,k}(x) \mid x \in F_{n,k}\} \cup \{X \sim F_{n,k}\}$ is an open cover of X. We show that for every $x \in X$, there exist natural numbers n and k such that the family $\mathcal{O}_{n,k}$ is a point-star $\dot{F}$-refinement of $\mathcal{U}$ at x. Let $x \in X$. Then there exists $n \in \mathbf{N}$ such that the family $(\mathcal{V}_n)_x$ is finite. Let A be a finite set of ordinals such that $(\mathcal{V}_n)_x = \{V_n(\alpha) \mid \alpha \in A\}$ and let $h = |A|$. Then $x \in F_{n,h}$. For each $\alpha \in A$, there exists $k(\alpha) \in \mathbf{N}$ such that the family $\mathcal{W}_{n,k(\alpha)}(\alpha)$ is point-finite at x. For each $\alpha \in A$, let $\mathcal{R}_\alpha = \{W \in (\mathcal{W}_{n,k(\alpha)}(\alpha))_x \mid W \subset U$ for some $U \in \mathcal{U}\}$ and let $\mathcal{U}_\alpha$ be a finite subfamily of $(\mathcal{U})_x$ such that the family $\mathcal{R}_\alpha$ is a partial refinement of the family $\mathcal{U}_\alpha$. Then $\mathcal{U}' = \bigcup_{\alpha \in A} \mathcal{U}_\alpha$ is a finite subfamily of $\mathcal{U}$ and $x \in \bigcap \mathcal{U}'$. Let $k = \max(\{h\} \cup \{k(\alpha) \mid \alpha \in A\})$. We show that $St(x, \mathcal{O}_{n,k}) \subset \bigcup \mathcal{U}'$.

Let $O \in \mathcal{O}_{n,k}$ be such that $x \in O$. Since $x \in F_{n,h} \subset F_{n,k}$, we see that there is $y \in F_{n,k}$ such that $O = O_{n,k}(y)$. Since $O_{n,k}(y) \subset V_n(\alpha(y,n))$, we have $x \in V_n(\alpha(y,n))$, that is, $\alpha(y,n) \in A$. Let $k' = k(\alpha(y,n))$. Then $k' \leqq k$ and it follows that $O_{n,k}(y) \subset W_{n,k'}(y)$. The set $W_{n,k'}(y)$ is contained in some set of the family $\{U_\beta | \beta \leq \alpha(n,y)\}$. Since $x \in O_{n,k}(y) \subset W_{n,k'}(y)$, it follows that $W_{n,k'}(y) \in \mathcal{R}_{\alpha(y,n)}$. Consequently, the set $W_{n,k'}(y)$ is contained in some set of the family $\mathcal{U}'$. It follows that $O_{n,k}(y) \subset \cup\mathcal{U}'$. We have shown that $St(x,\mathcal{O}_{n,k}) \subset \cup\mathcal{U}'$; hence the family $\mathcal{O}_{n,k}$ is a point-star $\dot{F}$-refinement of $\mathcal{U}$ at x. Arranging the open covers of the countable collection $\{\mathcal{O}_{n,k} | n \in \mathbf{N}\}$ in a sequence, it follows from the foregoing that we get a point-star $\dot{F}$-refining sequence for $\mathcal{U}$. This completes the proof of the inductive step.

By a well-known result of set theory, every monotone cover of X has a subcover which is well-ordered by set inclusion. If $\mathcal{L}$ is such a "well-monotone" cover of X, then $\cap\mathcal{L}' \in \mathcal{L}'$ for each $\mathcal{L}' \subset \mathcal{L}$; hence $\mathcal{L}$ is interior-preserving. Consequently, it follows from Theorem 4.1 that a topological space is submetacompact if every interior-preserving open cover of the space has a θ-sequence of open refinements. To be able to utilize this observation, we need the following lemmas.

Lemma **4**.2. *The following conditions are mutually equivalent for an interior-preserving open cover* $\mathcal{U}$ *of* X*:*

(i) $\mathcal{U}^F$ *has a σ-closure-preserving closed refinement.*

(ii) $\mathcal{U}^F$ *has a point-star refining sequence by interior-preserving open covers.*

(iii) $\mathcal{U}$ has a point-star $\dot{F}$-refining sequence by interior-preserving open covers.

Proof. (i) ⇒ (iii): Let $\mathcal{F} = \bigcup_{n\in\mathbf{N}} \mathcal{F}_n$ be a closed refinement of $\mathcal{U}^F$ such that for each $n \in \mathbf{N}$, the family $\mathcal{F}_n$ is closure-preserving. For all $n \in \mathbf{N}$ and $x \in X$, let $W_n(x) = [\cap(\mathcal{U})_x] \cap [X \sim \cup(\mathcal{F}_n \sim (\mathcal{F}_n)_x)]$. For every $n \in \mathbf{N}$, the families $\{\cap\mathcal{U}' | \mathcal{U}' \subset \mathcal{U}\}$ and $\{X \sim \cup\mathcal{F}' | \mathcal{F}' \subset \mathcal{F}_n\}$ are interior-preserving and open and it follows that the cover $\mathcal{W}_n = \{W_n(x) | x \in X\}$ of X has these same properties. To show that $\langle \mathcal{W}_n \rangle$ is a point-star $\dot{F}$-refining sequence for $\mathcal{U}$, let $x \in X$. There exists $n \in \mathbf{N}$, $F \in \mathcal{F}_n$ and a finite subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $x \in F \subset \cup\mathcal{U}'$. Let $\mathcal{U}'' = (\mathcal{U}')_x$. We show that $St(x,\mathcal{W}_n) \subset \cup\mathcal{U}''$. Let $y \in X$ be such that $x \in W_n(y)$. Then $x \notin \cup(\mathcal{F}_n \sim (\mathcal{F}_n)_y)$ and it follows, since $x \in F \in \mathcal{F}_n$, that $y \in F$. Consequently, there exists $U \in \mathcal{U}'$ such that $y \in U$. We have $x \in W_n(y) \subset \cap(\mathcal{U})_y \subset U$ and so $U \in \mathcal{U}''$. We have shown that $St(x,\mathcal{W}_n) \subset \cup\mathcal{U}''$. It follows from the foregoing that $\langle \mathcal{W}_n \rangle$ is a point-star $\dot{F}$-refining sequence for $\mathcal{U}$.

(iii) ⇒ (ii). Trivial.

(ii) ⇒ (i). Let $\langle \mathcal{V}_n \rangle$ be a point-star refining sequence for $\mathcal{U}^F$ such that each $\mathcal{V}_n$ is an interior-preserving open cover of X. For all $n \in \mathbf{N}$ and $\mathcal{U}' \subset \mathcal{U}$, let $F_n(\mathcal{U}') = \{x \in X | St(x,\mathcal{V}_n) \subset \cup\mathcal{U}'\}$. Note that for each $n \in \mathbf{N}$, if $x \in X$ and $y \in \cap(\mathcal{V}_n)_x$, then $(\mathcal{V}_n)_x \subset (\mathcal{V}_n)_y$ and hence $St(x,\mathcal{V}_n) \subset St(y,\mathcal{V}_n)$; it follows that $x \in F_n(\mathcal{U}')$ whenever the set $F_n(\mathcal{U}')$ intersects the neighborhood $\cap(\mathcal{V}_n)_x$ of x. For each $n \in \mathbf{N}$, let $\mathcal{F}_n = \{F_n(\mathcal{U}') | \mathcal{U}' \subset \mathcal{U}$ and $\mathcal{U}'$ finite$\}$. It follows from the foregoing that for each $n \in \mathbf{N}$, the family $\mathcal{F}_n$ is closed and closure-preserving. It is easily seen that the family $\bigcup_{n\in\mathbf{N}} \mathcal{F}_n$ is a refinement of the family $\mathcal{U}^F$.

Lemma 4.3. *Let* $\mathcal{U}$ *be an open cover of a submetacompact space* X. *Then there exists a σ-closure-preserving closed cover* $\mathcal{F}$ *of* X *such that for each* $x \in X$, *there exists* $F \in \mathcal{F}$ *and a finite subfamily* $\mathcal{U}'$ *of* $(\mathcal{U})_x$ *such that* $x \in F \subset \cup\mathcal{U}'$.

Proof. Let $\langle \mathcal{V}_n \rangle$ be a θ-sequence of open refinements of $\mathcal{U}$. For all $n \in \mathbf{N}$ and $k \in \mathbf{N}$, denote by $H_{n,k}$ the closed subset $\{x \in X \mid |(\mathcal{V}_n)_x| \leq k\}$ of X. For all $n \in \mathbf{N}$ and $\mathcal{V} \subset \mathcal{V}_n$, let $F_n(\mathcal{V}) = X \sim \cup(\mathcal{V}_n \sim \mathcal{V})$. For every $n \in \mathbf{N}$, let $\mathcal{F}_n = \{F_n(\mathcal{V}) \mid \mathcal{V} \subset \mathcal{V}_n$ and $\mathcal{V}$ finite$\}$ and further, for every $k \in \mathbf{N}$, let $\mathcal{F}_{n,k} = \mathcal{F}_n | H_{n,k}$. For all $n \in \mathbf{N}$ and $k \in \mathbf{N}$, the family $\mathcal{V}_n | H_{n,k}$ is a point-finite, and hence interior-preserving, open cover of the subspace $H_{n,k}$ of X; it follows that the family $\mathcal{F}_{n,k}$ is closed and closure-preserving in $H_{n,k}$ and hence in X, since $H_{n,k}$ is a closed subspace of X. We show that the σ-closure-preserving closed family $\mathcal{F} = \cup\{\mathcal{F}_{n,k} \mid n \in \mathbf{N}$ and $k \in \mathbf{N}\}$ has the property required in the lemma. Let $x \in X$. Since $\langle \mathcal{V}_n \rangle$ is a θ-sequence, there exist $n \in \mathbf{N}$ and $k \in \mathbf{N}$ such that $x \in H_{n,k}$. Let $\mathcal{U}'$ be a finite subfamily of $(\mathcal{U})_x$ such that $(\mathcal{V}_n)_x$ is a partial refinement of $\mathcal{U}'$. If we let $F = F_n((\mathcal{V}_n)_x) \cap H_{n,k}$, then $x \in F \in \mathcal{F}_{n,k}$ and it is easily seen that $F \subset \cup\mathcal{U}'$.

We are now ready to prove the main result of this section.

Theorem 4.4. *The following conditions are mutually equivalent for a topological space:*

(i) The space is submetacompact.

(ii) Every interior-preserving directed open cover of the

space has a point-star refining sequence by interior-preserving open covers of the space.

(iii) Every interior-preserving directed open cover of the space has a σ-closure-preserving closed refinement.

(iv) Every directed open cover of the space has a σ-closure-preserving closed refinement.

Proof. It follows from Lemma 4.3 that (i) ⇒ (iv). That (iv) ⇒ (iii) is obvious. The implication (iii) ⇒ (ii) follows from Lemma 4.2 using the observation that a cover $\mathcal{L}$ of X is directed iff $\mathcal{L}^F$ refines $\mathcal{L}$. To see that (ii) ⇒ (i) holds, assume that X satisfies (ii). Then it follows from Lemma 4.2 that every interior-preserving open cover of X has a point-star $\dot{F}$-refining sequence by interior-preserving open covers. Further, it follows by Remark 3.2.2 that every interior-preserving open cover of X has a θ-sequence of open refinements. Finally, it follows from the remark made after Theorem 4.1 that X is submetacompact.

As an easy consequence of Theorem 4.4, we have the following result, which extends a result of J. Chaber ([10]) and answers a question of J. Boone and R. Hodel ([4] and [15]).

Corollary 4.5. *A continuous image of a submetacompact space under a closed mapping is submetacompact.*

Proof. Let f be a closed and continuous mapping from a submetacompact space X onto a space Y. To show that the space Y is submetacompact, let $\mathcal{U}$ be a directed open cover of Y. Then the family $\mathcal{V} = \{f^{-1}(U) \mid U \in \mathcal{U}\}$ is a directed open cover of X. By Theorem 4.4, the cover $\mathcal{V}$ has a refinement $\bigcup_{n \in \mathbf{N}} \mathcal{F}_n$, where each $\mathcal{F}_n$ is a closure-preserving family of closed subsets of X. It is easily seen that for each $n \in \mathbf{N}$, the family $\mathcal{K}_n = \{f(F) | F \in \mathcal{F}_n\}$ is closed and closure-preserving in Y. Moreover, the family $\bigcup_{n \in \mathbf{N}} \mathcal{K}_n$ is a refinement of the cover $\mathcal{U}$. The conclusion now follows from Theorem 4.4.

By the remark following Proposition 3.5, the Locally Finite Sum Theorem holds for submetacompactness; note that this result can also be easily proved using Theorem 4.4.

Using Theorems 2.3 and 4.4, one can derive several characterizations for paracompactness, metacompactness and subparacompactness; for example, the characterizations given for metacompactness and paracompactness in Theorems 3.1 and 3.4 of [17] can be derived in this way. For subparacompact spaces, we get the following characterization.

Theorem 4.6. *A topological space is subparacompact iff every interior-preserving open cover of the space has a σ-closure-preserving closed refinement.*

Proof. Since every locally finite family is closure-preserving, the condition is necessary. To prove sufficiency, assume that every interior-preserving open cover of X has a σ-closure-preserving closed refinement. It follows from Theorem 4.4 that X is submetacompact. To complete the proof, let $\mathcal{F}$ be a discrete family of closed subsets of X and for each $F \in \mathcal{F}$, let $U(F) = X \sim \bigcup(\mathcal{F} \sim \{F\})$. Then the family $\mathcal{U} = \{U(F) | F \in \mathcal{F}\}$ is an interior-preserving open cover of X. The open cover $\mathcal{U}$ has a σ-closure-preserving closed refinement and it follows from a remark made after Lemma 3.9 that $\mathcal{U}$ has a point-star refining sequence by semi-open covers of X. The foregoing and the proof of Theorem 3.8 show that X is discretely subexpandable. The conclusion now follows from Theorem 2.3.

Leaving out "interior-preserving" from the above theorem, we have a result of D. Burke ([5]).

5. Submetacompactness, $\Sigma^{\#}$-Spaces and Strict p-Spaces

We now apply some results of the preceding section to obtain characterizations of submetacompactness in the classes of $\Sigma^{\#}$-spaces and strict p-spaces.

We need some definitions to state our first result. Recall that a family $\mathcal{N}$ of subsets of X is a *network* at a subset A of X provided that whenever U is an open subset of X such that $A \subset U$, then $A \subset N \subset U$ for some $N \in \mathcal{N}$. The space X is a *strong* $\Sigma^{\#}$*-space* (a $\Sigma^{\#}$*-space*) if X has a σ-closure-preserving closed cover $\mathcal{F}$ such that for each $x \in X$, the set $C(x) = \cap(\mathcal{F})_x$ is (countably) compact and the family $\mathcal{F}$ is a network at the set C(x) ([24]; it is an easy consequence of Lemma 1.6 of [25] that the above definition and those given in [24], [28] and [7] are all equivalent). A topological space is *isocompact* ([3]) if every countably compact closed subspace of the space is compact.

Theorem 5.1. *The following conditions are mutually equivalent for a topological space* X*:*

(i) X *is a strong* $\Sigma^{\#}$*-space.*

(ii) X *is an isocompact* $\Sigma^{\#}$*-space.*

(iii) X *is a submetacompact* $\Sigma^{\#}$*-space.*

Proof. By a result of [33], every submetacompact space is isocompact; consequently, we have (iii) $\Rightarrow$ (ii). That

(ii) $\Rightarrow$ (i) follows directly from the definitions. To prove that (i) $\Rightarrow$ (iii), assume that X is a strong $\Sigma^{\#}$-space. Let the cover $\mathcal{F}$ and the sets C(x), x $\in$ X, be as in the definition above, and let $\mathcal{C} = \{C(x) \mid x \in X\}$. Since the cover $\mathcal{C}$ of X consists of compact subsets of X, this cover is a refinement of every directed open cover of X. Since $\mathcal{F}$ is a network at each set of the cover $\mathcal{C}$, it follows that every directed open cover of X has a refinement consisting of sets of $\mathcal{F}$. Consequently, X satisfies condition (iv) of Theorem 4.4 and it follows from that theorem that X is submetacompact.

Before applying the above result to the theory of strict p-spaces, we use it to obtain a characterization of σ-spaces ([28]). We need the following auxiliary result.

Lemma 5.2. *Let* X *be a submetacompact space with a* G_δ*-diagonal. Then* X *has a* σ*-closure-preserving closed cover* $\mathcal{K}$ *such that* $\bigcap(\mathcal{K})_x = \{x\}$ *for each* x $\in$ X.

Proof. Since X has a G_δ-diagonal, it follows from a well-known result that X has a sequence $\langle \mathcal{U}_n \rangle$ of open covers such that $\bigcap_{n\in\mathbf{N}} St(x,\mathcal{U}_n) = \{x\}$ for each x $\in$ X. By Lemma 4.3 there exists for each n $\in \mathbf{N}$ a σ-closure-preserving closed cover $\mathcal{K}_n$ of X such that for each x $\in$ X, we have x $\in$ K $\subset$ $St(x,\mathcal{U}_n)$ for some K $\in \mathcal{K}_n$. Clearly, the family $\mathcal{K} = \bigcup_{n\in\mathbf{N}} \mathcal{K}_n$ has the properties required in the lemma.

Proposition 5.3. *A* T_1*-space is a* σ*-space iff it is a* $\Sigma^{\#}$*-space with a* G_δ*-diagonal.*

Proof. Necessity of the condition is known (see e.g. [7], Theorem 2.4.1). To prove sufficiency, assume that X is a $\Sigma^{\#}$-space with a G_δ-diagonal. It follows from Corollary 2.A of [9] that X is isocompact. By Theorem 5.1, X is submetacompact. The conclusion that X is a σ-space now follows from Lemma 5.2 and Corollary 2.2 (ii) of [28].

For strict p-spaces ([2]), we employ the following characterization ([8]): X is a strict p-space iff X is a Tychonoff-space and X has a sequence $\langle \mathcal{U}_n \rangle$ of open covers such that for each $x \in X$, the set $K(x) = \bigcap_{n \in \mathbf{N}} St(x, \mathcal{U}_n)$ is compact and the family $\{St(x, \mathcal{U}_n) \mid n \in \mathbf{N}\}$ is a network at the set K(x). It is an open question, whether every strict p-space is submetacompact (for some partial answers to the question, see [12]); we now use the result of Theorem 5.1 to obtain a translation of this question into another form.

Theorem 5.4. *A strict* p-*space is submetacompact iff it is a* $\Sigma^{\#}$-*space.*

Proof. Since every strict p-space is isocompact (see e.g. [12], Lemma 3), it follows from Theorem 5.1 that a strict p-space is submetacompact if it is a $\Sigma^{\#}$-space. To prove the converse, let X be a submetacompact strict p-space. Let the open covers $\mathcal{U}_n$, $n \in \mathbf{N}$, and the sets K(x), $x \in X$, be as in the above characterization of a strict p-space. By Lemma 4.3, there exists for each $n \in \mathbf{N}$ a σ-closure-preserving closed cover $\mathcal{F}_n$ of X such that for each $x \in X$, we have $x \in F \subset St(x, \mathcal{U}_n)$ for some $F \in \mathcal{F}_n$. Let $\mathcal{F}$ be the family consisting of all finite intersections of sets of the family $\bigcup_{n \in \mathbf{N}} \mathcal{F}_n$. We show that the family $\mathcal{F}$ has the properties required in the definition of a $\Sigma^{\#}$-space. It is easily seen that $\mathcal{F}$ is a σ-closure-preserving closed cover of X. For each $x \in X$, the set $C(x) = \cap(\mathcal{F})_x$ is compact, since this set is closed and it is contained in the compact set $K(x)$. To verify that $\mathcal{F}$ is a network at each set $C(x)$, let $x \in X$ and let U be an open subset of X such that $C(x) \subset U$. The family $\{U\} \cup \{X \sim F | F \in (\mathcal{F})_x\}$ is an open cover of X and hence there exists a finite subfamily $\mathcal{F}'$ of $(\mathcal{F})_x$ such that the compact set $K(x)$ is covered by the family $\{U\} \cup \{X \sim F | F \in \mathcal{F}'\}$. Then $U \cup (X \sim \cap\mathcal{F}')$ is an open set containing $K(x)$ and hence there exists $n \in \mathbf{N}$ such that $St(x, \mathcal{U}_n) \subset U \cup (X \sim \cap\mathcal{F}')$. Let $F \in \mathcal{F}_n$ be such that $x \in F \subset St(x, \mathcal{U}_n)$ and let $\mathcal{F}'' = \mathcal{F}' \cup \{F\}$. Then $F \subset U \cup (X \sim \cap\mathcal{F}')$ and it follows that $\cap\mathcal{F}'' \subset U$. Since $\mathcal{F}'' \subset (\mathcal{F})_x$, we have $C(x) \subset \cap\mathcal{F}''$. Moreover, $\mathcal{F}$ is closed under finite intersections and hence the set $\cap\mathcal{F}''$ is in $\mathcal{F}$. It follows from the foregoing that $\mathcal{F}$ is a network at the set $C(x)$.

Corollary 5.5. *An orthocompact strict* p-*space is submetacompact.*

Proof. Proposition 3.1 of [14] and Theorem 5.4.

6. An Open Question

We close this paper by stating what is perhaps the most important open question in the theory of submetacompact spaces. Note that an affirmative answer to the following question would imply that every strict p-space is submetacompact.

Question. Is a topological space submetacompact if every directed open cover of the space has a point-star refining sequence by (semi-) open covers?

In light of Lemma 3.9, this question is a restatement of Problem 2.7 of [19].

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Added in proof: for an important early characterization of submetacompactness, by J. M. Worrell Jr., see Notices Amer. Math. Soc. 14 (1967), p. 555; Worrell's result can be obtained as a corollary to Theorem 3.2 above.